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The four mirror Sagnac ring

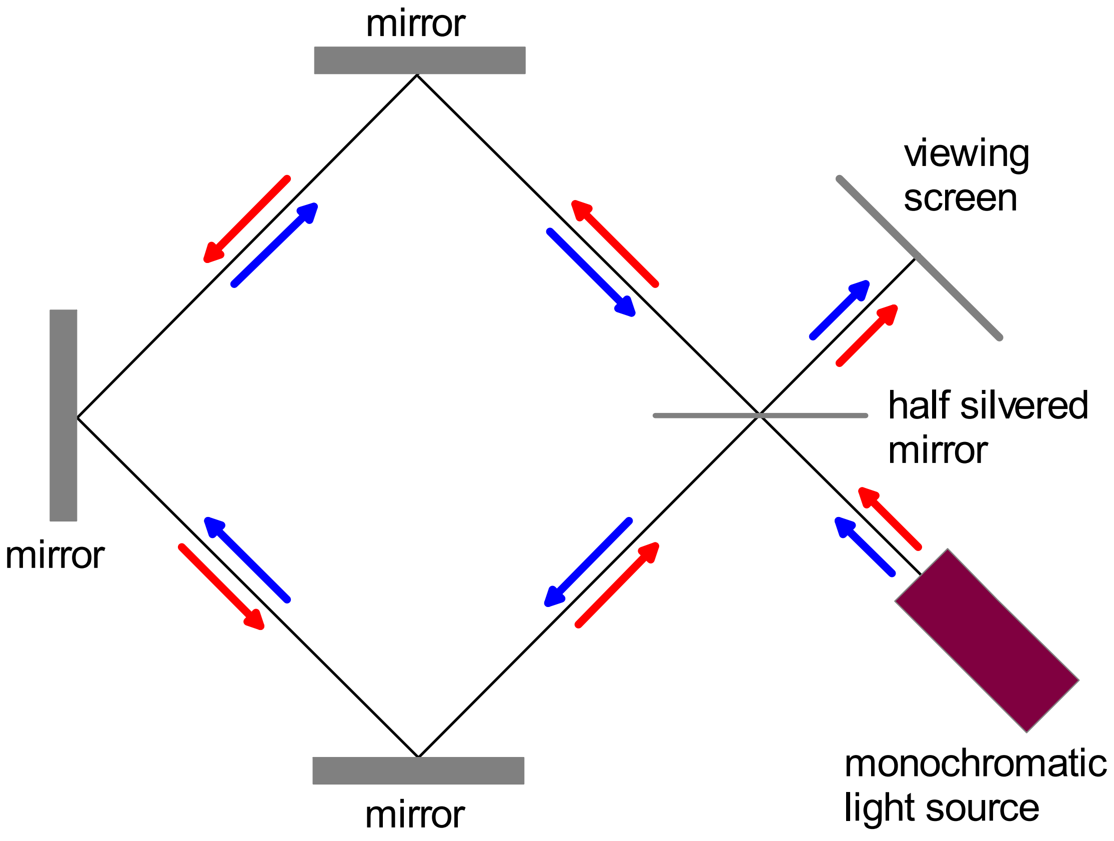

Fig. 1. The four mirror Sagnac interferometer

Fig. 1 shows a four mirror Sagnac interferometer. The light beam from the monochromatic source must be slightly divergent. If a laser is used, it may be necessary to place a diffuser lens in front of it. One part of the beam from the source passes through the half silvered mirror (red arrows) and is reflected off the three other mirrors back to the half silvered mirror. Part of the beam will pass through the half silvered mirror and hit the screen. This beam has gone around the ring in the anticlockwise direction.
Another part of the beam from the source will be reflected off the half silvered mirror, and will be reflected around the ring in the clockwise direction (blue arrows). Part of this beam will be reflected off the half silvered mirror and hit the screen. The light from the two beams will form an interference pattern on the screen. The pattern will depend on the collimation of the mirrors, but it will usually be a pattern with bright and dark fringes. If the central parts of the two beams are in phase, there will be a bright fringe at the centre of the screen.

When the Sagnac ring is rotating, the two beams will be slightly deflected in opposite directions. But since the beams are divergent, they will still overlap. The parts of the beams that hit at the centre of the screen will have hit each mirror in the centre. If these parts of the beams are out of phase, the bright fringe will be offset to one side.

We will calculate what the Special Theory of Relativity and the Ritz Emission Theory predict the phase difference will be when the interferometer is rotating.

There are two ways of calculating the phase difference:

1. The difference in transit time for the two beams can be calculated. The phase difference is then $\Delta\varphi = (2\pi c/\lambda)\cdot\Delta t$. The transit time is the time a plane of equal phase uses to move from the half silvered mirror, around the ring and back to the same mirror.
2. The number of wavelengths in the two beams can be compared. The phase difference is then $\Delta\varphi = 2\pi\cdot\Delta N$ where ΔN is the difference in the number of wavelengths in the two beams.

Even if both methods necessarily must give the same result, we will calculate the predictions both ways.

In all cases the calculation will be made in the non rotating inertial frame where the centre of the ring is stationary. All speeds and wavelengths will in the following be referred to this frame of reference.

The predictions calculated with the difference in transit times

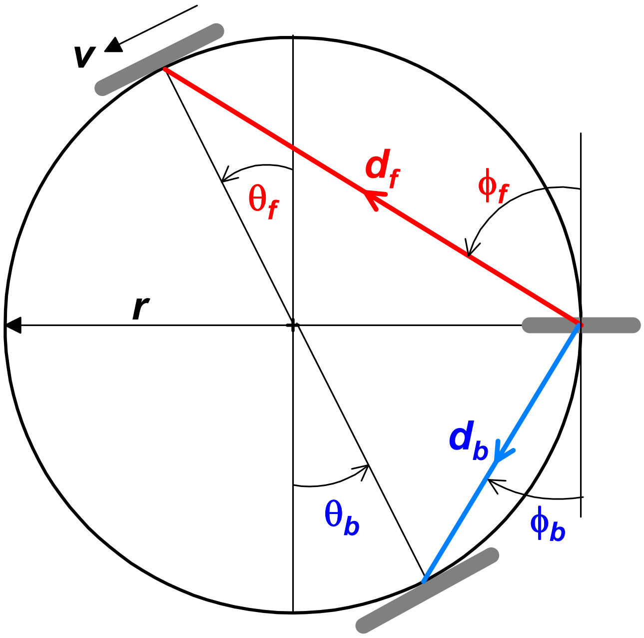

Fig. 2

Given that the centres of the mirrors are a distance r from the centre of the interferometer. The interferometer is rotating in the anticlockwise direction with a peripheral speed v. Since all the four parts of the beams between the mirrors will be equal, we can consider the first part of each beam, as shown in fig. 2.
Here are:

ϕ_f : the angle of the central beam emitted in the direction of rotation

ϕ_b : the angle of the central beam emitted in the direction opposite to the rotation

d_f : the path length of the central beam emitted in the direction of the rotation

d_b : the path length of the central beam emitted in the direction opposite to the rotation

T_f : the transit time of the central beam emitted in the direction of the rotation

T_b : the transit time of the central beam emitted in the direction opposite to the rotation

θ_f : the angle a mirror moves during the time T_f

θ_b : the angle a mirror moves during the time T_b

We have:

$$\mathrm{d_f} = \sqrt{\left(\mathrm{r} + \mathrm{r} \cdot \sin(\theta_\mathrm{f})\right)^2 + \left(\mathrm{r} \cdot \cos(\theta_\mathrm{f})\right)^2} = \mathrm{r} \cdot \sqrt{2\left(1 + \sin(\theta_\mathrm{f})\right)}$$

$$\theta_\mathrm{f} = \tfrac{\mathrm{v}}{\mathrm{r}} \cdot \mathrm{T_f}$$

$$\mathrm{d_f} = \mathrm{r} \cdot \sqrt{2\left(1 + \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_f})\right)} \tag{1}$$

$$\mathrm{d_b} = \sqrt{\left(\mathrm{r} - \mathrm{r} \cdot \sin(\theta_\mathrm{b})\right)^2 + \left(\mathrm{r} \cdot \cos(\theta_\mathrm{b})\right)^2} = \mathrm{r} \cdot \sqrt{2\left(1 - \sin(\theta_\mathrm{b})\right)}$$

$$\theta_\mathrm{b} = \tfrac{\mathrm{v}}{\mathrm{r}} \cdot \mathrm{T_b}$$

$$\mathrm{d_b} = \mathrm{r} \cdot \sqrt{2\left(1 - \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_b})\right)} \tag{2}$$

$$\cos\left(\phi_\mathrm{f}\right) = \frac{\mathrm{r} + \mathrm{r} \cdot \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_f})}{\mathrm{d_f}} = \frac{1 + \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_f})}{\sqrt{2\left(1 + \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_f})\right)}} = \tfrac{1}{\sqrt{2}} \cdot \sqrt{1 + \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_f})} \tag{3}$$

$$\cos\left(\phi_\mathrm{b}\right) = \frac{\mathrm{r} - \mathrm{r} \cdot \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_b})}{\mathrm{d_b}} = \frac{1 - \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_b})}{\sqrt{2\left(1 - \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_b})\right)}} = \tfrac{1}{\sqrt{2}} \cdot \sqrt{1 - \sin(\tfrac{\mathrm{v}}{\mathrm{r}}\mathrm{T_b})} \tag{4}$$

Prediction of the Special Theory of Relativity

According to this theory, the speed of light is c in the non rotating inertial frame.

We can then write:

$$c\cdot T_f = d_f = r\cdot\sqrt{2\left(1+\sin(\tfrac{v}{r}T_f)\right)}$$
$$c\cdot T_b = d_b = r\cdot\sqrt{2\left(1-\sin(\tfrac{v}{r}T_b)\right)}$$

These transcendental equations are not easy to solve. However, since the Sagnac effect is a first order effect in v/c, first order approximations will be satisfactory.

$$c\cdot T_f = r\cdot\sqrt{2\left(1+\sin(\tfrac{v}{r}T_f)\right)} \approx \sqrt{2}\cdot r\left(1+\frac{v}{2r}T_f\right) \qquad T_f \approx \frac{\sqrt{2}\cdot r}{c-\frac{1}{\sqrt{2}}v} \tag{5}$$

$$c\cdot T_b = r\cdot\sqrt{2\left(1-\sin(\tfrac{v}{r}T_b)\right)} \approx \sqrt{2}\cdot r\left(1-\frac{v}{2r}T_b\right) \qquad T_b \approx \frac{\sqrt{2}\cdot r}{c+\frac{1}{\sqrt{2}}v} \tag{6}$$

$$\Delta t = 4\left(T_f - T_b\right) \approx 4\left(\frac{\sqrt{2}\cdot r}{c-\frac{1}{\sqrt{2}}v} - \frac{\sqrt{2}\cdot r}{c+\frac{1}{\sqrt{2}}v}\right) = \frac{8rv}{c^2-\frac{1}{2}v^2} = \frac{8rv}{c^2\cdot\left(1-\frac{1}{2}\left(\frac{v}{c}\right)^2\right)} \approx \frac{8rv}{c^2}$$

Inserting the area enclosed by the light beam $A = 2r^2$ and the angular velocity $\omega = \frac{v}{r}$ yields:

$$\Delta t \approx \frac{4A\omega}{c^2}$$

The predicted phase difference is thus: $\Delta\varphi = \frac{2\pi c\cdot\Delta t}{\lambda} \approx \frac{8\pi A\omega}{\lambda\cdot c}$.

This is in accordance with the experimentally verified equation for a Sagnac ring.

Prediction of the Ritz Emission Theory

According to this theory, the speed of light is c relative to the source, and velocities transform according to the Galilean transform. The speed of the two light beams will then be:

Forward beam:

$$c_f = \sqrt{\left(c\cdot\sin\left(\phi_f\right)\right)^2 + \left(c\cdot\cos\left(\phi_f\right)+v\right)^2} = c\cdot\sqrt{1+\frac{2v}{c}\cdot\cos\left(\phi_f\right)+\frac{v^2}{c^2}}$$

$$c_f = c\cdot\sqrt{1+\frac{\sqrt{2}\cdot v}{c}\cdot\sqrt{1+\sin(\tfrac{v}{r}T_f)}+\frac{v^2}{c^2}} \tag{7}$$

Backward beam:

$$c_b = \sqrt{\left(c \cdot \sin\left(\phi_b\right)\right)^2 + \left(c \cdot \cos\left(\phi_b\right) - v\right)^2} = c \cdot \sqrt{1 - \frac{2v}{c} \cdot \cos\left(\phi_b\right) + \frac{v^2}{c^2}}$$

$$c_b = c \cdot \sqrt{1 - \frac{\sqrt{2} \cdot v}{c} \cdot \sqrt{1 - \sin(\tfrac{v}{r} T_b)} + \frac{v^2}{c^2}} \qquad (8)$$

Forward transit time:

$$c_f \cdot T_f = d_f$$

$$c \cdot \sqrt{1 + \frac{\sqrt{2} \cdot v}{c} \cdot \sqrt{1 + \sin(\tfrac{v}{r} T_f)} + \frac{v^2}{c^2}} \cdot T_f = r \cdot \sqrt{2\left(1 + \sin(\tfrac{v}{r} T_f)\right)}$$

Backward transit time:

$$c_b \cdot T_b = d_b$$

$$c \cdot \sqrt{1 - \frac{\sqrt{2} \cdot v}{c} \cdot \sqrt{1 - \sin(\tfrac{v}{r} T_b)} + \frac{v^2}{c^2}} \cdot T_b = r \cdot \sqrt{2\left(1 - \sin(\tfrac{v}{r} T_b)\right)}$$

These transcendental equations are not easy to solve. However, since the Sagnac effect is a first order effect in v/c, first order approximations will be satisfactory.

$$d_f = r \cdot \sqrt{2\left(1 + \sin(\tfrac{v}{r} T_f)\right)} \approx r \cdot \sqrt{2}\left(1 + \tfrac{v}{2r} T_f\right)$$

$$d_b = r \cdot \sqrt{2\left(1 - \sin(\tfrac{v}{r} T_b)\right)} \approx r \cdot \sqrt{2}\left(1 - \tfrac{v}{2r} T_b\right)$$

$$c_f = c \cdot \sqrt{1 + \frac{\sqrt{2} \cdot v}{c} \cdot \sqrt{1 + \sin(\tfrac{v}{r} T_f)} + \frac{v^2}{c^2}} \approx c \cdot \left(1 + \frac{\sqrt{2}}{2} \cdot \frac{v}{c}\right)$$

$$c_b = c \cdot \sqrt{1 - \frac{\sqrt{2} \cdot v}{c} \cdot \sqrt{1 - \sin(\tfrac{v}{r} T_b)} + \frac{v^2}{c^2}} \approx c \cdot \left(1 - \frac{\sqrt{2}}{2} \cdot \frac{v}{c}\right)$$

$$c \cdot \left(1 + \frac{\sqrt{2}}{2} \cdot \frac{v}{c}\right) \cdot T_f \approx r \cdot \sqrt{2}\left(1 + \frac{v}{2r} T_f\right) \qquad T_f \approx \sqrt{2} \cdot \frac{r}{c} \qquad (9)$$

$$c \cdot \left(1 - \frac{\sqrt{2}}{2} \cdot \frac{v}{c}\right) T_b \approx r \cdot \sqrt{2}\left(1 - \frac{v}{2r} T_b\right) \qquad T_b \approx \sqrt{2} \cdot \frac{r}{c} \qquad (10)$$

$$\Delta t = 4\left(T_f - T_b\right) \approx 0$$

The predicted phase difference is thus: $\Delta\varphi = 0$

Since the experimentally verified equation for a Sagnac ring is $\Delta\varphi \approx 8\pi A w / \lambda c$, the Sagnac experiment falsifies the Ritz Emission Theory.

The predictions calculated by comparing number of wavelengths

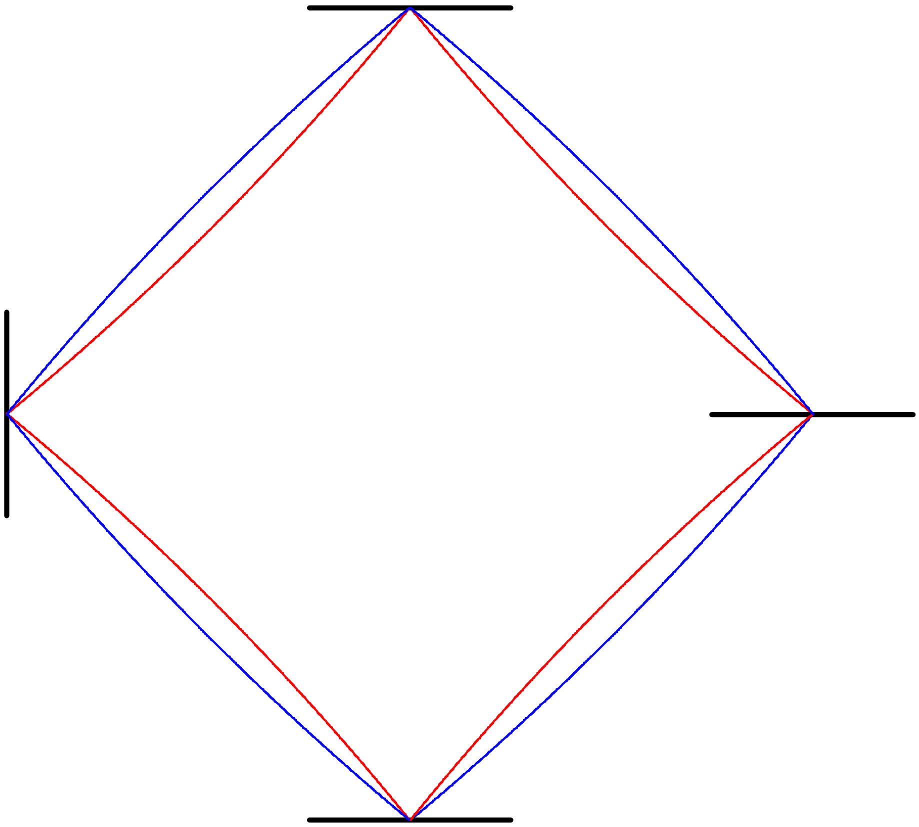

Fig.3

Fig. 3 shows an instant image of the central part of the beams, drawn in the non rotating inertial frame. The beams are slightly curved because the different parts of the beams were emitted at different angles as measured in the non rotating frame. The beam going with the rotation (red curve) is slightly concave, while the beam going in the opposite direction (blue curve) is slightly convex. At the mirrors, the angles to the mirrors are ϕ_f and ϕ_b respectively, as defined in fig. 2.

From equation (3),(4),(5),(6),(9) and (10) we find that a first order approximation of the cosines of the angles can be written:

$$\cos(\phi_f) \approx \frac{\sqrt{2}}{2} + \frac{v}{2c} \qquad (11) \qquad \cos(\phi_b) \approx \frac{\sqrt{2}}{2} - \frac{v}{2c} \qquad (12)$$

Note that the first order approximations of the angles are the same for the Special Theory of Relativity and the Ritz Emission Theory.

A first order Taylor expansion of $\cos(\phi)$ where $\phi = \frac{\pi}{4} \pm \delta$ yields:

$$\cos\left(\frac{\pi}{4} + \delta\right) \approx \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\delta \qquad (13) \qquad \cos\left(\frac{\pi}{4} - \delta\right) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\delta \qquad (14)$$

Comparing (11) and (14), (12) and (13) shows that first order approximations of the angles are:

$$\phi_f \approx \frac{\pi}{4} - \delta \qquad \phi_b \approx \frac{\pi}{4} + \delta \qquad \text{where } \delta = \frac{\sqrt{2}}{2} \cdot \frac{v}{c}$$

Going back to fig.3, we will see that the curved beams must be sections of circles. We will find the lengths of these sections.

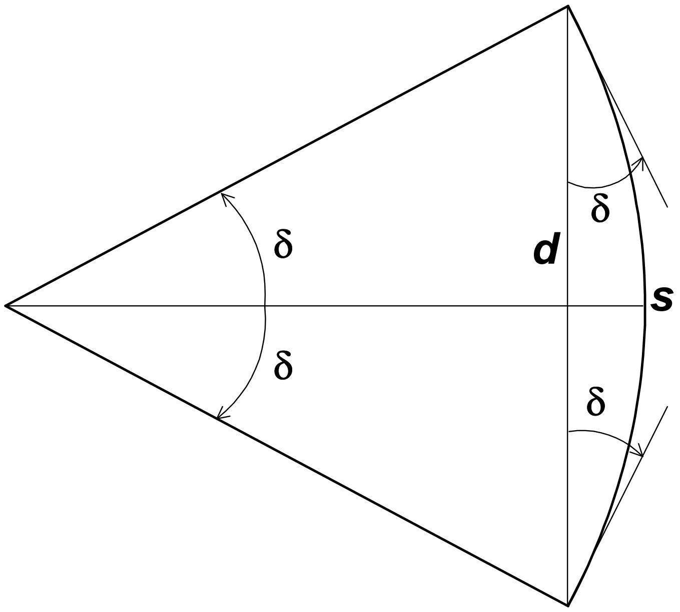

Fig. 4

From fig.4, we find:

$$s = d \cdot \frac{\delta}{\sin(\delta)} \quad \text{where d is the distance between the mirrors, } d = \sqrt{2} \cdot r$$

A Taylor expansion of $\sin(\delta)$ yields:

$$\sin(\delta) \approx \delta - \frac{\delta^3}{6}$$

Hence:

$$s \approx d\left(1 + \frac{\delta^2}{6}\right) \approx \sqrt{2}r\left(1 + \frac{1}{12}\frac{v^2}{c^2}\right)$$

This shows that there will be no measurable effect of the curvature of the beams.

We can thus consider the lengths of the beams to be $4\sqrt{2} \cdot r$

Prediction of the Special Theory of Relativity

Since the source is moving, the wavelengths of the beams will according to the Special Theory of Relativity be Doppler shifted.

The wavelength in the forward beam will then be:

$$\lambda_f = \frac{1-\cos(\phi_f)\cdot\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\lambda \approx \frac{1-\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{v}{c}\right)\cdot\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\lambda \approx \left(1-\frac{\sqrt{2}}{2}\cdot\frac{v}{c}\right)\cdot\lambda$$

The wavelength in the backward beam will be:

$$\lambda_b = \frac{1+\cos(\phi_b)\cdot\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\lambda \approx \frac{1+\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{v}{c}\right)\cdot\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\lambda \approx \left(1+\frac{\sqrt{2}}{2}\cdot\frac{v}{c}\right)\cdot\lambda$$

The number of wavelengths in the forward beam becomes:

$$N_f = \frac{4\cdot\sqrt{2}\cdot r}{\lambda_f} = \frac{4\cdot\sqrt{2}\cdot r}{\left(1-\frac{\sqrt{2}}{2}\cdot\frac{v}{c}\right)\cdot\lambda}$$

The number of wavelengths in the backward beam becomes:

$$N_b = \frac{4\cdot\sqrt{2}\cdot r}{\lambda_b} = \frac{4\cdot\sqrt{2}\cdot r}{\left(1+\frac{\sqrt{2}}{2}\cdot\frac{v}{c}\right)\cdot\lambda}$$

$$\Delta N = N_f - N_b \approx \frac{4\cdot\sqrt{2}\cdot r}{\left(1-\frac{\sqrt{2}}{2}\cdot\frac{v}{c}\right)\cdot\lambda} - \frac{4\cdot\sqrt{2}\cdot r}{\left(1+\frac{\sqrt{2}}{2}\cdot\frac{v}{c}\right)\cdot\lambda} = \frac{8rv}{\lambda c\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)} \approx \frac{8rv}{\lambda c}$$

Inserting the area enclosed by the light beam $\mathrm{A} = 2\mathrm{r}^2$ and the angular velocity $\omega = \frac{\mathrm{v}}{\mathrm{r}}$ yields:

$$\Delta N \approx \frac{4A\omega}{\lambda c}$$

The predicted phase difference is thus: $\Delta\varphi = 2\pi\cdot\Delta\mathrm{N} \approx \dfrac{8\pi\mathrm{A}\omega}{\lambda\cdot\mathrm{c}}$

Prediction of the Ritz Emission Theory

According to the Ritz Emission Theory wavelengths are not Doppler shifted. This is a consequence of the Galilean transform.

The number of wavelengths in the two beams is therefore:

$$N_f = \frac{4 \cdot \sqrt{2} \cdot r}{\lambda}$$

$$N_b = \frac{4 \cdot \sqrt{2} \cdot r}{\lambda}$$

$$\Delta N = N_f - N_b = 0$$

The predicted phase difference is thus: $\Delta\varphi = 2\pi \cdot \Delta N = 0$

Conclusion

We have shown that the Special Theory of Relativity predicts the phase difference between the beams in a rotating four mirror Sagnac ring to be $\approx 8\pi A\omega/\lambda c$.
Since this is in accordance with experimental evidence within the precision of the measurement, the Sagnac experiment confirms the Special Theory of Relativity.

We have shown that the Ritz Emission Theory predicts the phase difference between the beams in a rotating four mirror Sagnac ring to be less than a small number which is consistent with zero, and is less than the precision of any practical measurement.
Since experimental evidence shows the phase difference to be $\approx 8\pi A\omega/\lambda c$, the Sagnac experiment falsifies the Ritz Emission Theory.