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What the Special Theory of Relativity predicts for a FOG

We will calculate what the Special Theory of Relativity predicts the phase difference will be between the contra moving light rays in a rotating fibre optic gyro.

There are two ways of calculating this phase difference:

1. The difference in transit time for the two beams can be calculated. The phase difference is then $\Delta\varphi = 2\pi\nu \cdot \Delta t$ where ν is the frequency of the light.
   The transit time is the time a plane of equal phase uses to move around the ring.
2. The number of wavelengths in the two beams can be compared. The phase difference is then $\Delta\varphi = 2\pi \cdot \Delta N$ where ΔN is the difference in the number of wavelengths in the two beams.

Even if the two methods necessarily must give the same result, we will calculate the predictions both ways.

Given:

- Let the fibre be a single circular loop with radius r.
- Let the peripheral speed of the fibre be v as measured in the inertial frame where the centre of the ring is stationary.
- Let n be the index of refraction in the fibre.
- Let c be the speed of light in vacuum.

The prediction calculated with the difference in transit times

At any point on the fibre, the speed of light will be c/n as measured in an instantly co-moving inertial frame.

We transform this speed to the non rotating inertial frame where the fibre is moving at the speed v:

The beam going with the rotation:

$$c_f = \frac{\frac{c}{n} + v}{1 + \frac{\frac{c}{n} \cdot v}{c^2}} \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right)$$

The beam going in the opposite direction:

$$c_b = \frac{\frac{c}{n} - v}{1 - \frac{\frac{c}{n} \cdot v}{c^2}} \approx \frac{c}{n} - v\left(1 - \frac{1}{n^2}\right)$$ (Second order terms and higher in v/c ignored.)

The time t_f to go around the circular loop with radius r is for the beam going with the rotation:

$$t_f \cdot c_f = 2\pi r + v \cdot t_f$$

$$t_f = \frac{2\pi r}{c_f - v}$$

For the other beam:

$$t_b \cdot c_b = 2\pi r - v \cdot t_b$$

$$t_b = \frac{2\pi r}{c_b - v}$$

$$\Delta t = t_f - t_b = 2\pi r\left(\frac{1}{c_f - v} - \frac{1}{c_b + v}\right) = 2\pi r\left(\frac{c_b - c_f + 2v}{(c_f - v)(c_b + v)}\right)$$

$$\Delta t = 2\pi r\left(\frac{\left(\frac{c}{n} - v\left(1-\frac{1}{n^2}\right)\right) - \left(\frac{c}{n} + v\left(1-\frac{1}{n^2}\right)\right) + 2v}{\left(\left(\frac{c}{n} + v\left(1-\frac{1}{n^2}\right)\right) - v\right)\left(\left(\frac{c}{n} - v\left(1-\frac{1}{n^2}\right)\right) + v\right)}\right) = \frac{2\pi r\left(\frac{2v}{n^2}\right)}{\left(\frac{c}{n} - \frac{v}{n^2}\right)\left(\frac{c}{n} + \frac{v}{n^2}\right)} = \frac{2\pi r\left(\frac{2v}{n^2}\right)}{\frac{c^2}{n^2}\left(1-\left(\frac{v}{nc}\right)^2\right)} = \frac{4\pi r v}{c^2\left(1-\left(\frac{v}{nc}\right)^2\right)}$$

We ignore second order terms in v/c, and get:

$$\Delta t = \frac{4\pi r v}{c^2}$$

We insert the area of the loop $A = \pi r^2$ and the angular velocity $\omega = \frac{v}{r}$ and get:

The phase difference $\Delta\varphi = 2\pi\nu \cdot \Delta t = \frac{2\pi c \cdot \Delta t}{\lambda} \approx \frac{8\pi A\omega}{\lambda \cdot c}$

Note that λ is the wavelength and c is the speed of the light in vacuum.

This is consistent with the experimentally verified equation for a fibre optic gyro (FOG).

Note that the index of refraction does not affect the phase difference.

The prediction calculated by comparing number of wavelengths

- Let ν be the frequency of the light source as measured in an instantly co-moving inertial frame.
- Let λ be the wavelength in vacuum of light with frequency ν, $\lambda = c/\nu$
- Let ν_f and λ_f be the frequency and wavelength of the ray that is moving with the rotation, as measured in the inertial frame where the centre of the ring is stationary.
- Let ν_b and λ_b be the frequency and wavelength of the ray that is moving in the opposite direction, as measured in the inertial frame where the centre of the ring is stationary.

The frequency of the beam going with the rotation will be Doppler shifted in the inertial frame:

$$\nu_f = \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{1-\dfrac{v}{c_f}} \cdot \nu = \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{1-\dfrac{v}{c_f}} \cdot \frac{c}{\lambda}$$

$$\lambda_f = \frac{c_f}{\nu_f} = \frac{c_f\left(1-\dfrac{v}{c_f}\right)}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \cdot \frac{\lambda}{c} = \frac{\lambda}{c} \cdot \frac{c_f - v}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

We ignore second order terms in v/c, and get:

$$\lambda_f \approx \frac{\lambda}{c} \cdot \left(c_f - v\right) = \frac{\lambda}{c} \cdot \left(\frac{c}{n} + v\left(1-\frac{1}{n^2}\right) - v\right) = \frac{\lambda}{n} \cdot \left(1-\frac{v}{nc}\right)$$

The frequency of the beam going in the other direction will be Doppler shifted in the inertial frame:

$$\nu_b = \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{1+\dfrac{v}{c_b}} \cdot \nu = \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{1+\dfrac{v}{c_b}} \cdot \frac{c}{\lambda}$$

$$\lambda_b = \frac{c_b}{\nu_b} = \frac{c_b\left(1+\dfrac{v}{c_b}\right)}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \cdot \frac{\lambda}{c} = \frac{\lambda}{c} \cdot \frac{c_b + v}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

We ignore second order terms in v/c, and get:

$$\lambda_b \approx \frac{\lambda}{c} \cdot \left(c_b + v\right) = \frac{\lambda}{c} \cdot \left(\frac{c}{n} - v\left(1-\frac{1}{n^2}\right) + v\right) = \frac{\lambda}{n} \cdot \left(1+\frac{v}{nc}\right)$$

The number of wavelengths in the beams will be:

$$N_f = \frac{2\pi r}{\lambda_f} \approx \frac{2\pi r}{\frac{\lambda}{n}\cdot\left(1-\frac{v}{nc}\right)} = \frac{2\pi rn}{\lambda\left(1-\frac{v}{nc}\right)}$$

$$N_b = \frac{2\pi r}{\lambda_b} \approx \frac{2\pi r}{\frac{\lambda}{n}\cdot\left(1+\frac{v}{nc}\right)} = \frac{2\pi rn}{\lambda\left(1+\frac{v}{nc}\right)}$$

The difference in the number of wavelengths will be:

$$\Delta N = N_f - N_b \approx \frac{2\pi rn}{\lambda\left(1-\frac{v}{nc}\right)} - \frac{2\pi rn}{\lambda\left(1+\frac{v}{nc}\right)} = \frac{4\pi rv}{\lambda c\left(1-\left(\frac{v}{nc}\right)^2\right)}$$

Ignoring second order terms and inserting the area of the ring $A = \pi r^2$ and the angular velocity of the ring $\omega = v/r$ yields:

$$\Delta N \approx \frac{4A\omega}{\lambda c}$$

The predicted phase difference is thus:

$$\Delta\varphi \approx \frac{8\pi Aw}{\lambda c}$$