

An Elementary Introduction to
Depth Two Characterizations
of Hopf-Galois Extensions

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Consider various properties of a finite degree field extension $E \supseteq F$; e.g. $\mathbb{Q}(\sqrt{a}) | \mathbb{Q}$ where $a \in \mathbb{Z}_+$.

1. A *separable extension* if there's an elt. $\sigma \in E \otimes_F E$ s.t. $e\sigma = \sigma e \forall e \in E$ and $\text{mult}(\sigma) = \sigma^1 \sigma^2 = 1$. E.g. $\sigma = \frac{1}{2a}(\sqrt{a} \otimes \sqrt{a} + a \otimes 1)$ is separability elt. for $\mathbb{Q}(\sqrt{a}) | \mathbb{Q}$.
2. A *normal extension* if E is the splitting field of a set of polynomials in $F[x]$; e.g. $\mathbb{Q}(\sqrt{a})$ is the splitting field of $x^2 - a$ over \mathbb{Q} .
3. A *Galois extension* if the group of automorphisms of E over F fixes precisely F : in symbols, $\text{End}_F E \supset G = \text{Gal}(E | F) = \{\alpha : E \xrightarrow{\cong} E : \alpha(fe) = f\alpha(e) \forall f \in F, e \in E\}$ and $E^G = F$, i.e. $\alpha(x) = x \forall \alpha \in G \Rightarrow x \in F$.

E.g. if a is square-free, $G = \{\text{id}, \text{conj}\}$ where $\text{conj}(r+s\sqrt{a}) = r-s\sqrt{a}$, so $\mathbb{Q}(\sqrt{a})^G = \mathbb{Q}$.

4. Separable or Galois extensions are *Frobenius extensions*, i.e. there's a trace map $T : E \rightarrow F$ which in Galois case is def. by $T(e) = \sum_{g \in G} \sigma(g)(e)$; e.g. $T(r + s\sqrt{a}) = 2r$ using id and conj.

Theorem (Steinitz, 1909). $E|F$ is Galois iff $E|F$ is a separable and normal extension.

All types of extensions above except normal extension have successful generalizations to non-commutative ring or algebra extensions (Frobenius, 1954; Separable, 1966: Hopf-Galois, 1981).

We propose the notion *depth two* as a non-commutative analog of normality.

Origins in C^* - and von Neumann algebras

Bratteli diagram for inclusion of fin. dim. C^* -algebras: e.g. $S_2 < S_3$ and their corresp. group \mathbb{C} -algebras is unitarily equiv. to inclusion mapping $(\lambda, \mu) \mapsto (\lambda, \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \mu)$

Inclusion matrix is induction-restriction table for irreducible characters:

$S_2 \leq S_3$	χ_1	χ_2	χ_3
ψ_1	1	1	0
ψ_2	0	1	1

Jones tower of von Neumann factors:

$$N \subset M \subset M_1 = \text{End } M_N \subset M_2 \subset \dots$$

(iteration by basic End construction). Basic construction of f.d. C^* -algebras has a mirror image Bratteli diagram.

Derived tower of centralizers or relative commutants:

$$C_N(N) \subset C_M(N) \subset C_{M_1}(N) \subset C_{M_2}(N) \subset \dots$$

Draw Bratteli diagrams of each inclusion : of depth n if at n 'th level it begins to repeat itself via reflections.

The notion depth two (D2) extends to any ring extension (Nikshych, Szlachanyi, L.K.)

Let $B \subseteq A$ be a subring with $1_B = 1_A$. $A|B$ is D2 if there's a split epimorphism

$$A \oplus \dots \oplus A \longrightarrow A \otimes_B A$$

of the natural A - B -bimodules (right D2) and B - A -bimodules (left D2).

Since $\text{Hom}(A, A \otimes_B A) \cong (A \otimes_B A)^B := T$
and $\text{Hom}(A \otimes_B A, A) \cong \text{End}_B A_B := S$
we have equiv. def. of D2 extension:

(left D2 quasibase) $\exists \beta_i \in S, t_i \in T$:

$$a \otimes a' = \sum_{i=1}^n t_i \beta_i(a) a'$$

(right D2 quasibase) $\exists \gamma_j \in S, u_j \in T$:

$$a \otimes a' = \sum_{j=1}^m a \gamma_j(a') u_j$$

Example. A normal subgroup of a finite group:
 $N \triangleleft G$. If $G = \coprod_{i=1}^n g_i N$, $A = \mathbb{C} G$ and $B = \mathbb{C} N$,
then $t_i = g_i \otimes g_i^{-1}$ and $\beta_i(\sum_{g \in G} a_g g) = \sum_{g \in g_i N} a_g g$
is a left D2 quasibase since for all $g \in G$

$$g \otimes_N 1 = \sum_{i=1}^n g_i \otimes g_i^{-1} \beta_i(g)$$

Quantum Groups replace groups with *Hopf algebras* and Galois extensions with Hopf-Galois extensions (1980's and 90's).

Group algebra $\mathbb{C}G$ is Hopf algebra with comultiplication $\Delta(g) = g \otimes g$, counit (scalar map) $\varepsilon(g) = 1$ and antipode $S(g) = g^{-1}$.

Hopf-Galois extension $A|B$ satisfies four properties:

1. action of a Hopf alg. H on A :
 $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$, $h \triangleright 1_A = \varepsilon(h)1_A$,
e.g. G acts like an automorphism of A , or
a Lie alg. acts like a derivation on A .
2. invariants or fixed point subalgebra $A^H = B$, i.e. $h \triangleright x = \varepsilon(h)x$, $\forall h \in H \Leftrightarrow x \in B$.

3. End $A_B \cong A \# H$ where
 $(a \# h)(b \# k) = a(h_{(1)} \triangleright b) \# h_{(2)}k$ and
 $a \# h \mapsto \lambda(a)(h \triangleright -)$
4. A is a finitely generated projective right B -module: one can find B -valued module maps f_i and $a_i \in A$ such that
 $a = \sum_{i=1}^n a_i f_i(a) \quad \forall a \in A.$

E.g. a Galois extension $E|F$ is Hopf-Galois since $|G| = [E : F].$

A Hopf-algebra-free characterization of Hopf-Galois extension for certain algebra extensions:

Theorem (Nikshych-Szlachanyi-L.K.) Let $A|B$ be a Frobenius extension of F -algebras with trivial centralizer $C_A(B) = F1_A.$ Then $A|B$ is Hopf-Galois iff $A|B$ is D2 and right balanced.

PF. (\Rightarrow) dualize action to coaction

$\rho : A \rightarrow A \otimes H$, coinvariant B where $\rho(b) = b \otimes 1_H$.

Then $A \otimes_B A \cong A \otimes H$ via Galois mapping

$a \otimes a' \mapsto aa'_{(0)} \otimes a'_{(1)}$ as A - B -bimodules, so

$A \otimes_B A \cong \bigoplus^n A$ where $n = \dim H$, whence $A|B$

is right D2 extension. Left D2 extension as

well by utilizing an alternative Galois mapping

$a \otimes a' \mapsto a_{(0)}a' \otimes a_{(1)}$.

Module A_B is balanced if $\text{End}_{\mathcal{E}} A = \rho(B)$ where

$\mathcal{E} := \text{End } A_B$. Since $A \hookrightarrow \mathcal{E}$, $f \in \text{End}_{\mathcal{E}} A$ deter-

mined by H -invariant $f(1) \in B$.

(\Leftarrow) Somewhat in the spirit of Tannakian re-

construction theorems, one must build dual

Hopf algebras on algebras $S = \text{End}_B A_B$ and

$T = (A \otimes_B A)^B$ where S and T are nondegen-

erately paired by $\langle \alpha | t \rangle = \alpha(t^1)t^2$ and T has mul-

tiplication $tt' = t'^1 t^1 \otimes t^2 t'^2$ and unit $1_T = 1 \otimes 1$.

Depth two extension admits bialgebroid struc-

tures on S and T over centralizer. Here we

assume it's trivial, so we have bialgebras, or Hopf algebras without antipodes. But Frobenius structure admits integrals, which in turn admit antipodes for both S and T . Action of S on A by $\alpha \triangleright a = \alpha(a)$. S -Invariants $B = A^S$, since A_B balanced, and $\text{End } A_B \cong A \# S$ since $A|B$ is D2.

CONCLUDING REMARKS

Since the mid-90's Hopf algebras are being replaced by weak Hopf algebras as more flexible quantum gauge groups or fusion algebras in conformal field theories.

E.g. a groupoid algebra is a weak Hopf algebra. A groupoid G is a finite category with invertible arrows. E.g. $G_n := \{1, \dots, n\}$, source $s(e_{ij}) = i$, target $t(e_{ij}) = j$, groupoid algebra FG where $gh = \begin{cases} 0 & \text{if } s(g) \neq t(h) \\ g \circ h & \text{if } s(g) = t(h) \end{cases}$

E.g. $FG_n = M_n(F)$, where e_{ij} are matrix units!

Weak Hopf algebra structure: comultiplication $\Delta(g) = g \otimes g$, counit $\varepsilon(g) = 1$ and antipode $S(g) = g^{-1}$ for all $g \in G$. E.g. $S(e_{ij}) = e_{ji}$ transpose of matrices.

Differs from Hopf algebra in that

1. $\Delta(1) \neq 1 \otimes 1$, e.g. $\Delta(1_{M_n}) = \sum_{i=1}^n e_{ii} \otimes e_{ii}$;
2. $\varepsilon(gh) \neq \varepsilon(g)\varepsilon(h)$, e.g. $\varepsilon(e_{12}e_{34}) = 0$ but $\varepsilon(e_{12})\varepsilon(e_{34}) = 1$.
3. $S(e_{ij})e_{ij} = e_{jj} \neq \varepsilon(e_{ij})1$

We have a notion of weak Hopf-Galois extension based on Galois corings (T. Brzezinski).

More study in terms of alternative characterizations, as a Frobenius extension, in terms of smash products and actions rather than coactions and some evidence I have would likely yield a weak version of the Hopf-free D2 characterization of Galois extension:

Proposed Theorem. Let $A|B$ be a Frobenius extension of algebras where the centralizer is a separable algebra. Then $A|B$ is a weak Hopf-Galois extension $\iff A|B$ is of depth two and right balanced.