Galois theory, anchor maps and Hopf algebroids

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INTRODUCTION: a quantization story

Hopf algebroid	Hopf algebra
base algebra	base field
groupoid	group
Lie algebroid	Lie algebra
bundle	fiber
quantum groupoid	quantum group
D2 algebra extension	D2 alg. ext. with trivial centralizer

Why Galois theory?

Principal bundle $P \times G \longrightarrow P \longrightarrow Y$

Pass to function algebras.
Dualize $P \times G \longrightarrow P \times_Y P$.
Free action ↔ Galois *iso*morphism

$$A \otimes_B A \to A \otimes H$$

Hopf-Galois extension $B \hookrightarrow A \to A \otimes H$.

Poisson Algebra: a commutative algebra with a bilinear Lie bracket $\{\,,\,\}$ satisfying Leibniz: $\{ef, g\} = e\{f, g\} + \{e, g\}f$.

Example. Polynomial algebra $\mathbb{R}\,[x, p]$ with $\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}$. Tensor product of Poisson algebras $\rightsquigarrow$ usual Poisson bracket in any even number of variables for polynomials.

Example. A Poisson manifold is a smooth manifold M where $C^\infty(M)$ is a Poisson algebra. Since $\{f, \cdot\}$ is a derivation on $C^\infty(M)$, there is a (Hamiltonian) vector field V_f s.t.

$$\{f, g\} = \langle V_f, dg\rangle$$

Example. A symplectic manifold is a manifold with a nondegenerate closed 2-form ω. Let V_f now be defined by $df(u) = \omega(u, V_f)$, all tangent vectors u. Then

$$\{f, g\} = \omega(V_f, V_g)$$

defines a Poisson bracket. E.g. $\omega = \sum_i dx_i \wedge dp_i$ is a symplectic form on $\mathbb{R}^{2n}$ whose Poisson algebra extends the polynomial algebra above. The cotangent bundle of any manifold is symplectic.

Quantization of Poisson algebras

Start: commutative algebra A

Aim: nontrivial associative algebra structure on ring of formal power series $A[[h]]$ where $A \cong A[[h]]/(h)$.

Proceed: multiplication $*$ given by $(f, g \in A)$
$f * g = fg + hc_1(f,g) + h^2 c_2(f,g) + \cdots$
where $c_i : A \otimes A \to A$, Hochschild 2-cochains.

Associativity condition: $(c_0(f,g) = fg = gf)$
$\sum_{i+j=n} c_j(c_i(e,f),g) = \sum_{p+q=n} c_p(e, c_q(f,g))$

Equivalently: $\partial c_1 = 0$, $\partial c_2 = \mathrm{Sq}\, c_1$, $\partial c_3 = [c_1, c_2]_G$, Gerstenhaber bracket...

Loosely: coboundaries $\leftrightarrow$ equivalences of $A[[h]]$ with itself, $H^2(A, A) \leftrightarrow$ infinitesimal deformations, and $H^3(A, A) \leftrightarrow$ obstructions to carrying out aim.

Suppose all obstructions are passed, e.g. $H^3(A, A) = 0$. Then $A[[h]]$ is alg. and A has a Poisson bracket: $\{f, g\} = c_1(f, g) - c_1(g, f)$.

Def. Call $(A[[h]], *)$ a quantization of A, and call the Poisson algebra $(A, \{,\})$ the quasiclassical limit of $(A[[h]], *)$.

Example. $A = \mathbb{R}\,[x, p]$ with $\{,\}$ above is quasiclassical limit of $(A[[h]], *)$ where

$$f * g = m(e^{\frac{1}{2}h(\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial p} - \frac{\partial}{\partial p}\otimes\frac{\partial}{\partial x})} f \otimes g)$$

Def. Lie bialgebra = Lie algebra $(\mathcal{G}, [,])$ with (cobracket) map $\delta : \mathcal{G} \to \Lambda^2\mathcal{G}$ satisfying a co-Jacobi identity and a cocycle condition.

Example. $sl_2(\mathbb{C})$ with basis $h = e_{11} - e_{22}$, $e = e_{12}$ and $f = e_{21}$. Relations $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. Cobracket:

$$\delta(e) = \frac{1}{2}e \wedge h,\ \delta(f) = \frac{1}{2}f \wedge h,\ \delta(h) = 0$$

Example. A Poisson-Lie group G is a Lie group and Poisson manifold w/ mult. $G \times G \to G$ a map of Poisson manifolds. Then its Lie algebra $T_eG = \mathcal{G}$ is a Lie bialgebra w $\delta = d\Pi$ where Π is Poisson bivector $G \to \Lambda^2\mathcal{G}$.

Quantum universal enveloping (QUE) algebras

Recall: Hopf algebra A is algebra, coalgebra w/ comultiplication and counit alg. homos. and antipode map which is anti-homo. and inverse to identity wrt convolution product on $\operatorname{End} A$. E.g. a group algebra, its dual and univ. env. alg. $U(\mathcal{G})$ of a Lie algebra $\mathcal{G}$.

A quantization of A is a Hopf algebra $A[[h]]$ quantizing A as an alg. and where its comultiplication
$\Delta = \Delta_0 + h\Delta_1 + h^2\Delta_2 + \cdots$ is an algebra homo. formed from coalg. cocycle Δ_1 on A which passes all obstructions in degree three coalg. cohomology.

A QUE algebra $U_h(\mathcal{G})$ has quasi-classical limit $U(\mathcal{G})$. Then $\delta = \Delta_1 - \Delta_1^{\mathrm{op}}$ is a cobracket making $\mathcal{G}$ a Lie bialgebra. $U_h(\mathcal{G})$ is a co-Poisson Hopf algebra.

Example. $U_h(sl_2(\mathbb{C}))$ gen. by E, F, H, relations $[H, E] = 2E$, $[H, F] = 2F$, $[E, F] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}$, coproduct $\Delta(E) = E \otimes e^{hH} + 1 \otimes E$, $\Delta(F) = F \otimes 1 + e^{-hH} \otimes F$, $\Delta(H) = H \otimes 1 + 1 \otimes H$, antipode $S(E) = -Ee^{-hH}$, $S(F) = -e^{hH}F$, $S(H) = -H$, counit $\varepsilon(E) = \varepsilon(F) = \varepsilon(H) = 0$.

Sometimes expressed in rational form with $q = e^h$, $K = e^{hH}$, a group-like element ($\Delta(K) = K \otimes K$, $\varepsilon(K) = 1$ and $S(K) = K^{-1}$).

GROUPOIDS. A groupoid G is a category whose objects form a set X, called the base set, and all of whose arrows are invertible. If we identify G with the arrows, there are two maps, called source and target, $\sigma, \tau : G \to X$. The identity section $\varepsilon : X \to G$ sends an x to id_x. The product on G is composition, defined only on composable pairs (g, h) where $\sigma(g) = \tau(h)$. Inversion $G \to G$ sends each arrow to its inverse. Various topological or smoothing conditions may be imposed on these sets and maps to define Lie groupoids, etc.

Examples. 1) A group is a groupoid over a singleton set X. (The isotropy group of any groupoid $\mathcal{G}$ is defined over an x in X. Orbit of x are all points that may be reached from x.)

2) The pair groupoid on any set X is $X \times X$ with the projection maps as source and target.

This may be viewed as the coarsest equivalence relation on X. Any equivalence relation corresponds to a subgroupoid.

3) Fundamental groupoid of a top. space X is $\Pi(X) = \{(x, [\gamma], y) | \gamma(0) = x, \gamma(1) = y \in X\}$, where fundamental group is an isotropy group at a base point and orbits are path components.

There are groupoids in group theory, Galois theory, Poisson geometry, Lie theory, foliations: [W].

Groupoid Algebra - Discrete version. Given finite groupoid G, extend by one element $0 \notin G$, provide $G \cup \{0\}$ with semigroup structure $gh = 0$ if $\tau(h) \neq \sigma(g)$ and $g0 = 0 = 0g$. Form the standard semigroup algebra. Example: full matrix algebra $M_n(\mathbb{C})$ where $e_{ij}e_{rs} = e_{is}\delta_{jr}$ for pair groupoid on $\{1, 2, \ldots, n\}$. $\exists$ groupoid C^*-algebra of a locally compact groupoid with Haar system on σ-fibers: [W].

Groupoid Action takes place on fiber spaces over same base space X. If fibers are vector spaces, we have a representation of groupoid.

The infinitesimal version of Lie groupoid:

Lie Algebroid. A Lie algebroid over a manifold X is a real vector bundle E over X together with a bundle map $\rho : E \to TX$, called the **anchor**, and a real Lie algebra bracket $[\,,\,]_E$ on sections $\Gamma(E)$ s.t.

1) the anchor induces a Lie algebra homomorphism $\Gamma(\rho) : \Gamma(E) \to \Gamma(TX)$.

2) for any $f \in C^\infty(X), V, W \in \Gamma(E)$ we have Leibniz: $[V, fW]_E = f[V, W]_E + (\rho(V) \cdot f)W$.

Examples. 1) A Lie algebra, where $X = \text{pt}$. 2) A bundle of Lie algebras over X, where $\rho = 0$. 3) The tangent bundle TX of a manifold X, where $\rho = \text{id}$. A special case of:
4) the normal bundle E along the identity section ε of a Lie groupoid can be made a Lie algebroid, where TX above is the Lie algebroid of $X \times X$ or $\Pi(X)$.
5) Lie algebroid of a symplectic manifold (X, ω): via nondegenerate 2-form ω a bundle isomorphism $T^*X \to TX$ which is anchor map and standard bracket on differential forms: e.g. $[df, dg] = d\{f, g\}$. Extends to Poisson manifolds [W].

"Alternative tangent bundle with peculiar differentiable structure where one carries out virtually all diffential-geometric constructions (e.g. de Rham cohomology, Lie bracket and connections)."

Quantization. A Lie algebroid A has a type of universal enveloping algebra UA [PX], a cocommutative Hopf algebroid. E.g. $A = TX$ then $UA =$ algebra of differential operators on X. Ping Xu shows how to deform the algebra UA to get noncocommutative QUE Hopf algebroid U_hA. The semiclassical limit of such a deformation is always a *Lie bialgebroid*, which is a special Lie algebroid-coalgebroid.

DEFINITION OF HOPF ALGEBROID w/ total algebra H and base algebra R [Lu, PX BoSz]:

1. H and R are (unital associative) k-algebras,

2. there are commuting algebra homomorphisms $R \xrightarrow{\sigma} H \xleftarrow{\tau} R^{\mathrm{op}}$ (called *source* and *target*, respectively) where for each $r, s \in R$ we have $\sigma(r)\tau(s) = \tau(s)\sigma(r)$,

3. fix the R-R-bimodule ${}_RH_R$ given by

   $$r \cdot h \cdot s = \sigma(r)\tau(s)h \quad (h \in H)$$

4. H has *comultiplication* $\Delta : H \to H \otimes_R H$ given by notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ where $\Delta(1_H) = 1_H \otimes_R 1_H$.

5. multiplicativity of Δ with technical pre-condition: for all $h, g \in H$, $r \in R$,

   $$h_{(1)}\tau(r) \otimes_R h_{(2)} = h_{(1)} \otimes_R h_{(2)}\sigma(r)$$

   $$\Delta(hg) = \Delta(h)\Delta(g)$$

6. an (anchor) map $\mu : H \to \mathrm{End}_k R$, an R-R-bimodule morphism wrt bimodule $r \cdot f \cdot r' = rf(-)r'$ and represent. of H on k-linear space R (so write $\mu(h)(r) := h \triangleright r$) satisfying:

7. $\sigma(h_{(1)} \triangleright r)h_{(2)} = h\sigma(r)$

8. $\tau(h_{(2)} \triangleright r)h_{(1)} = h\tau(r)$.

   Axioms above define a left R-bialgebroid H.

   Existence of an anchor map equivalent to existence of counit $\varepsilon : H \to R$ [BM] (set $\varepsilon(h) = h \triangleright 1_R$) s.t. $(H, R, \Delta, \varepsilon)$ is generalized coalgebra (over possibly noncommutative R), $\varepsilon(1_H) = 1_R$, and

   $$\varepsilon(gh) = \varepsilon(g\sigma(\varepsilon(h))) = \varepsilon(g\tau(\varepsilon(h)))$$

   A Hopf algebroid (with bijective antipode) is a bialgebroid s.t.

9. there is an anti-automorphism $S : H \to H$ such that

10. $S \circ \tau = \sigma$,

11. $\overline{S}(h_{(2)})_{(1)} \otimes_R \overline{S}(h_{(2)})_{(2)} h_{(1)} = \overline{S}(h) \otimes_R 1_H$, (where $S^{-1} = \overline{S}$)

12. $S(h_{(1)})_{(1)} h_{(2)} \otimes_R S(h_{(1)})_{(2)} = 1_H \otimes_R S(h)$ $(\forall\, h \in H)$

Example 1. Let $R = k =$ base field. Then Hopf algebra and Connes-Moscovici twisted Hopf algebras are Hopf algebroids.

Example 2. Let R be a separable algebra over a field. Then a weak Hopf algebra is a Hopf algebroid [EN].

DEPTH TWO EXTENSION. Alg. homo. or subalgebra $B \to A$ is right D2 if natural bimodules ${}_A A_B$ and ${}_A A \otimes_B A_B$ satisfy type of projectivity condition: $\exists$ split epi

$$ {}_A A \oplus \cdots \oplus A_B \longrightarrow {}_A A \otimes_B A_B $$

Equivalently, $\sum_j x\gamma_j(y)u_j^1 \otimes_B u_j^2 = x \otimes_B y$, all $x, y \in A$ for some $\gamma_j \in \text{End}\,{}_B A_B$, $u_j = u_j^1 \otimes u_j^2 \in (A \otimes_B A)^B$. (Left D2 similarly def.)

Example. Depth two subfactors, finite dimensional algebras; normal subgroups, Hopf subalgebras; depth one or centrally projective extensions, Galois and pseudo-Galois extensions over groups, Hopf algebra, weak Hopf algebras.

Theorem [KS]. The algebras End ${}_B A_B$ and $(A \otimes_B A)^B \cong \text{End}\,({}_A A \otimes_B A_A)$ are bialgebroids, fin. projective over $R = C_A(B) = \{a \in A | \forall b \in B, ba = ab\}$ and dual to one another.

PF. Left and right mult. & commuting maps, $R \xrightarrow{\lambda} E = \text{End}\,{}_B A_B \xleftarrow{\rho} R^{\text{op}}$.

Note $E \otimes_R E \xrightarrow{\cong} \text{Hom}_{B-B}(A \otimes_B A, A)$ via $\alpha \otimes_R \beta \mapsto \alpha \cup \beta$, cup product on relative Hochschild cochains (extends to n-cochains = cup product of 1-cochains). Define comult. $\Delta : E \to E \otimes_R E$ by $\Delta(\alpha)(x \otimes y) = \alpha(xy)$ for $x, y \in A$.

Anchor $\mu : E \to \text{End}\, R$ given by $\mu(\alpha)(r) = \alpha(r)$, i.e. restriction of natural action by eval. of $\text{End}\,{}_B A_B$ on A. Then counit $\varepsilon : H \to R$ is $\varepsilon(\alpha) = \alpha(1_A)$.

Duality between E and $T = (A \otimes_B A)^B$ given by R-val. nondegen. pairing $\langle \alpha, t^1 \otimes_B t^2 \rangle = \alpha(t^1)t^2$. Right bialgebroid structure on T given by mult. $tt' = {t'}^1 t^1 \otimes_B t^2 {t'}^2$, source $\sigma(r) = 1 \otimes_B r$, target $\tau(r) = r \otimes_B 1$, comult. $\Delta(t) = t^1 \otimes 1 \otimes t^2$ where $T \otimes_R T \cong (A \otimes_B A \otimes_B A)^B$, anchor $\mu(t)(r) = t^1 r t^2$, a generalized Miyashta-Ulbrich action of Hopf algebra on centralizer of Hopf-Galois extension, or quotient group conjugation action on centralizer of normal subgroup [KK]. Counit $\varepsilon : T \to R$, $\varepsilon(t) = t^1 t^2$.

Theorem [KS], [LK]. A is an algebra in the tensor category E-MOD, and a Galois extension over B if ${}_BA$ is balanced module.

PF. Two E-modules $V, W \rightsquigarrow E$-module $V \otimes_R W$ via $\alpha \cdot (v \otimes w) = (\alpha_{(1)} \cdot v) \otimes (\alpha_{(2)} \cdot w)$. Unit module is R via anchor representation.

E-module algebra: $x, y \in A$, $\alpha \in E = \mathrm{End}\,{}_BA_B$: $\alpha \triangleright x = \alpha(x)$, eval. action.
Measuring: $\alpha \triangleright (xy) = (\alpha_{(1)} \triangleright x)(\alpha_{(2)} \triangleright y)$.
A^E = Invariants = generalized simultaneous eigenvectors w/ eigenvalue $\alpha(1) \in R$, each $\alpha \in E$. E.g. $B \subseteq A^E$: $\alpha \triangleright b = \alpha(1) b$.

Module ${}_BA$ is balanced if $B \xrightarrow{\lambda} \mathrm{End}\, A_{\mathcal{E}}$ is onto, where $\mathcal{E} = \mathrm{End}\, {}_BA$. If so or $B = C_A(R)$, one shows $A^E = B$.

Galois Map $\beta : A \otimes_B A \xrightarrow{\cong} A \otimes_R T$ given by $\beta(a \otimes_B a') = \sum_j a\gamma_j(a') \otimes_R u_j$ in terms of D2 quasibase $u_j \in T$, $\gamma_j \in E$.

Galois map inverse: $a \otimes_R t \mapsto at^1 \otimes_B t^2$ since $\sum_j a\gamma_j(a')u_j = a \otimes_B a'$. ($A$ is dually a right T-comodule algebra [LK].)

When do E or T have an antipode? Suppose $A \,|\, B$ is Frobenius algebra in $B-$MOD$-B$, i.e. nondenerate B-B-bimodule map $F : A \to B$ w/ dual bases $x_i, y_i \in A$ s.t. $\mathrm{id}_A = \sum_j F(-x_j)y_j = \sum_j x_j F(y_j -)$. For example E has antipode

$$S(\alpha) = \sum_j x_j F(\alpha(y_i)-)$$

Below we summarize other Hopf algebroids arising as bialgebroids of depth two extensions.

D2 Ext. $A \,|\, B$	*Bialgebroid* End ${}_B A_B$ *or Dual* $(A \otimes_B A)^B$
Fin. proj. alg. A [Lu]	End A and A^e
Irred. Frob. ext. [KN1]	dual Hopf algs.
Frob. ext. w/ sep. cent. [KS]	dual weak Hopf algs. [KN2]
H-separable ext. [LK0]	Hopf algebroid R^e
Hopf-Galois ext. [LK1]	Lu Geometric Hopf algebroid
Pseudo-Galois ext. [LK2]	Connes-Mosc. Hopf algebroi

Example: pseudo-Galois extension $A \,|\, B$ [MM], satisfying as A-A-bimodules,

$$A \otimes_B A \oplus * \cong \oplus_{\sigma \in G} A_\sigma^N$$

where G is finite subgroup of algebra automorphisms of A fixing each element of B, and A_σ denotes the twisted A-module given by
$a \cdot x = a\sigma(x)$ for $\sigma \in G$.

By finding certain generalized Casimir elts., it turns out End ${}_B A_B$ is linearly generated by endomorphisms $\lambda(r) \circ \sigma \circ \rho(s)$ where $r, s \in R = C_A(B)$ and $\sigma \in G$. Note that for $r, s, u, v \in R$ and $\sigma, \tau \in G$:

$$\lambda(r)\sigma\rho(s) \circ \lambda(u)\tau\rho(v) = \lambda(r\sigma(u)) \circ \sigma\tau \circ \rho(v\tau^{-1}(s))$$

Thus there is an R-bialgebroid epimorphism $R^e \bowtie G \to$ End ${}_B A_B$ given by

$$r \otimes s \bowtie \sigma \longmapsto \lambda(r) \circ \sigma \circ \rho(s)$$

where multiplication on $R^e \bowtie G$ given by

$$(r \otimes s \bowtie \sigma)(u \otimes v \bowtie \tau) = r\sigma(u) \otimes v\tau^{-1}(s) \bowtie \sigma\tau$$

Via the depth two machinery, easy to write down Hopf algebroid structure more generally on $A^e \bowtie H$ where H is an involutive Hopf algebra with left H-module algebra A [LK2].

Hopf algebroid $A^e \bowtie H$ where H is Hopf algebra with antipode satisfying $S_H^2 = \mathrm{id}_H$ and A is left H-module algebra:

Multiplication: $(a \otimes b \bowtie h)(c \otimes d \bowtie k) =$
$a(h_{(1)} \cdot c) \otimes d(\tau_H(k_{(2)}) \cdot b) \bowtie h_{(2)}k_{(1)}$,

Identity: $1_A \otimes 1_A \bowtie 1_H$,

Source: $\lambda(a) = a \otimes 1_A \bowtie 1_H$,

Target: $\rho(a) = 1_A \otimes a \bowtie 1_H$,

A-A-Bimodule:
$a \cdot (b \otimes c \bowtie h) \cdot d = ab \otimes c(\tau_H(h_{(2)}) \cdot d) \bowtie h_{(1)}$

Comultiplication:
$\Delta(a\otimes b \bowtie h) = (a\otimes 1_A \bowtie h_{(1)})\otimes_A(1_A\otimes b \bowtie h_{(2)})$,

Anchor: $\mu(a \otimes b \bowtie h)(x) = a(h \cdot (xb))$,
Counit: $\varepsilon(a \otimes b \bowtie h) = a(h \cdot b)$,

Antipode: $S(a \otimes b \bowtie h) = b \otimes a \bowtie S_H(h)$.

Theorem [OP]. As Hopf algebroids,
$A^e \bowtie H \cong A \odot H \odot A$
the Connes-Moscovici Hopf algebroid via

$$a \otimes b \bowtie h \longmapsto a \odot h_{(1)} \odot h_{(2)} \cdot b$$

with inverse,

$$a \odot h \odot b \longmapsto a \otimes \tau_H(h_{(2)}) \cdot b \bowtie h_{(1)}$$

Remark. The anchor for Connes-Moscovici is $a \odot h \odot b \mapsto \lambda_a \circ \rho_b \circ (h \cdot -)$ as representation on A, whereas the anchor computed in the previous slide was $a \otimes b \bowtie h \mapsto \lambda_a \circ (h \cdot -) \circ \rho_b$. A short computation shows:

$$\lambda_a \circ (h \cdot -) \circ \rho_b = \lambda_a \circ \rho_{h_{(2)} \cdot b} \circ (h_{(1)} \cdot -),$$

which lifts to the isomorphism above.

Some questions and problems related to depth two:

1. Chirality Problem.
   Is there a left D2 algebra extension that is not right D2?
2. Normality Problem.
   Is a D2 Hopf subalgebra normal?
3. Galois Correspondence Problem.
   Find new conditions under which D2 subextensions correspond to Hopf subalgebroids.
4. Inverse Galois Problem.
   Given a f.g. projective left bialgebroid, is it the endomorphism ring of a D2 extension?

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