Invitation to Depth Two

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Depth two extends the notion of normal subgroup and is a noncommutative analog of normal field extension.

DEFINITION OF D2 EXTENSION [KS] Denote a ring or algebra homomorphism $B \to A$ by $A\,|\,B$ and consider its natural bimodule ${}_BA_B$. We say $A\,|\,B$ is right D2 if $\exists$ split epi

$$\oplus^n {}_AA_B \longrightarrow {}_AA \otimes_B A_B \qquad (n \in \mathbb{Z}_+)$$

Equivalently in $A \otimes_B A$

$$x \otimes y = \sum_{j=1}^{n} x\gamma_j(y)u_j^1 \otimes u_j^2$$

for n endomorphisms $\gamma_j \in \mathrm{End}\,{}_BA_B$ and n B-central tensors $u_j = u_j^1 \otimes u_j^2 \in (A \otimes_B A)^B$ (called a right D2 quasibase).

Similarly a left D2 ext. $A\,|\,B$ w/ left D2 quasibase satisfies an equation

$$x \otimes y = \sum_{i=1}^{m} t_i^1 \otimes t_i^2 \beta_i(x) y$$

Example 1. HOPF-GALOIS EXT'S.
H = bialgebra of $\dim H = n$, $A =$ an H-comodule algebra w/ coaction $A \to A \otimes H$ s.t. $B = A^{\mathrm{co\,H}}$ and *i*somorphism

$$\beta : A \otimes_B A \to A \otimes H, \quad \beta(a \otimes a') = aa'_{(0)} \otimes a'_{(1)}$$

Then ${}_A A \otimes_B A_B \cong \oplus^n {}_A A_B$.
H has an antipode [Sch], so $\beta'(a \otimes a') = a_{(0)} a' \otimes a_{(1)}$ is isomorphism, and $A|B$ is left D2 as well.

Example 2. NORMAL SUBGROUP $H \lhd G$ has D2 group algebras. Let $g_1, \ldots, g_n$ be coset representatives, $t_i := g_i \otimes g_i^{-1}$ satisfies $ht_i = t_i h$ $\forall h \in H$, β_i = canon. proj. onto $g_i H$, satisfies $\beta_i(hgh') = h\beta_j(g)h'$ and $\sum_i g_i \otimes g_i^{-1} \beta_i(g) = g \otimes 1$ $\forall g \in G$

OBS. If $A \,|\, B$ is D2, then the centralizer $R = C_A(B)$ is a normal subring in A: i.e. $\forall$ ideal $I \subset A$

$$(I \cap R)A = A(I \cap R) \tag{1}$$

Reason: $\forall r \in R \cap I$, $\exists A \otimes_B A \to I$, $x \otimes_B y \mapsto xry$, apply to $1 \otimes_B a = \sum_j \gamma_j(a) u_j^1 \otimes_B u_j^2$ to get

$$ra = \sum_j \gamma_j(a) u_j^1 r u_j^2 \in A(I \cap R)$$

Similarly $ar = \sum_i t_i^1 r t_i^2 \beta_i(a) \in (I \cap R)A$, whence eq. (1).

OBS. Using the similar equations

$$a_{(1)} \otimes S(a_{(2)}) a_{(3)} = a \otimes 1$$

$$a_{(1)} S(a_{(2)}) \otimes a_{(3)} = 1 \otimes a$$

for $a \in$ a Hopf algebra H w/ antipode S, we see normal Hopf subalgebra $K \subseteq H$ (i.e. every $x \in H$ satisfying $S(x_{(1)}) K x_{(2)} \subseteq K$ and $x_{(1)} K S(x_{(2)}) \subseteq K$) is a normal subring. Converse follows from $HK^+ = K^+ H$ characterization (where $K^+ = \ker \varepsilon \cap K$).

Consequence: A D2 Hopf subalgebra $K \subseteq H$ is normal in case $K = C_H(C_H(K))$ and $C_H(K)$ is a Hopf subalgebra.

PF. $C_H(K)$ is normal subring $\Rightarrow$ normal Hopf subalgebra $\Rightarrow$ Hopf-Galois extension $\Rightarrow$ D2 $\Rightarrow$ double centralizer is normal.

PROBLEM. Is a D2 Hopf subalgebra always a normal Hopf subalgebra?
More evidence:
1. [KK] If $H \leq G$ is a subgroup of a finite group, $\mathbb{C}\,H \subseteq \mathbb{C}\,G$ is D2 $\Leftrightarrow H \triangleleft G$
using Mackey character theory.

2. If Hopf subalgebra $K \subseteq H$ is Hopf-W-Galois ext. where H, K, W are fin. dim. Hopf algebras, then $K \subseteq H$ is normal. PF. Define $H \to W$ by $h \mapsto \varepsilon(h_{(0)})h_{(1)}$, w/ kernel $= HK^+ = K^+H$ by Galois-Schneider theory for H-module coalgebra quotients [HJS].

This last example is not so far-fetched since a free D2 extension like $H \supseteq K$ is Galois in an extended sense.

Theorem [NK, SK, LK] An algebra extension $A \,|\, B$ is a right T-Galois ext. for some left fin. projective right R-bialgebroid T $\Leftrightarrow$ it is right D2 and balanced.

Explanation. $\Rightarrow$ is rather easy.
($\Leftarrow$) Let $T := (A \otimes_B A)^B \overset{\cong}{\longleftarrow} \text{End}\,({}_A A \otimes_B A_A)$ via $F \mapsto F(1 \otimes 1)$, which induces the multiplication on T:

$$tu = u^1 t^1 \otimes_B t^2 u^2, \quad 1_T = 1_A \otimes 1_A$$

T is a right bialgebroid over the centralizer R:

1. groupoid set-up: $R \xrightarrow{s_R} T \xleftarrow{t_R} R^{\text{op}}$ via $s_R(r') = 1 \otimes_B r'$ and $t_R(r) = r \otimes_B 1$ satisfying $s_R(r')t_R(r) = t_R(r)s_R(r')$.

2. bimodule ${}_RT_R = T_{s_R,t_R}$: $r \cdot t \cdot r' = rt^1 \otimes t^2 r'$. ${}_RT$ is finite projective from quasibase eq. $t^1 \otimes_B t^2 = \sum_j t^1 \gamma_j(t^2) u_j$.

3. (T, Δ, ε) is R-coring where comultiplication $\Delta_T : T \to T \otimes_R T \cong (A \otimes_B A \otimes_B A)^B$, $t_{(1)} \otimes t_{(2)} = t^1 \otimes_B 1 \otimes_B t^2$ and counit $\varepsilon_T(t) = t^1 t^2 \in R$.

4. bialgebroid properties: $\Delta(1_T) = 1_T \otimes_R 1_T$, $\Delta(tu) = \Delta(t)\Delta(u)$, $\varepsilon(1_T) = 1_R$, and right properties $s_R(r)t_{(1)} \otimes t_{(2)} = t_{(1)} \otimes t_R(r)t_{(2)}$, $\varepsilon(tu) = \varepsilon(t_R(\varepsilon(t))u) = \varepsilon(s_R(\varepsilon(t))u)$.

A is a right T-comodule algebra: coaction $\delta : A \to A \otimes_R T$, $a_{(0)} \otimes_R a_{(1)} = \sum_j \gamma_j(a) \otimes_R u_j$ satisfies $\delta(1_A) = 1_A \otimes_R 1_T$ and $\delta(xy) = \delta(x)\delta(y)$.

Coinvariants $A^{\mathrm{co}\,\mathrm{T}} = \{x \in A | \, x_{(0)} \otimes_R x_{(1)} = x \otimes 1\} = B$ where $\subseteq$ follows from A_B balanced.

Finally $A \otimes_R T$ is Galois A-coring with bimodule $a(a' \otimes t)a'' = aa'a''_{(0)} \otimes_R ta''_{(1)}$, comult. $A \otimes \Delta_T$ and counit $A \otimes \varepsilon_T$. Grouplike elt. $1_A \otimes 1_T$ and isomorphism $A \otimes_R T \cong A \otimes_B A$ via
$a \otimes_R t \mapsto at^1 \otimes_B t^2$ w/ inverse

$$a \otimes_B a' \longmapsto \sum_j a\gamma_j(a') \otimes_R u_j = aa'_{(0)} \otimes a'_{(1)}$$

the Galois isomorphism. $\square$

Note. Entwining map $\psi : T \otimes_R A \to A \otimes_R T$, $\psi(t \otimes_R a) = a_{(0)} \otimes_R ta_{(1)} \stackrel{\cong}{\longmapsto} t^1 \otimes_B t^2 a$ is bijective if $A \,|\, B$ is also left D2. See [BB] for a Chern-Galois character $K_0(\mathcal{C}) \to HC_{2n}(B|D)$.

Example. An H-extension $A \,|\, B$ w/ split injective Galois (A-B-)mapping $\beta : A \otimes_B A \hookrightarrow A \otimes H$ is D2, therefore T-Galois.
If β is $\cong$, then $T^{\text{op}} \cong R \# H^{\text{op}}$ is a Lu-Hopf algebroid where h acts by $r \triangleleft h = a^1 r a^2$ and $1 \otimes h = \beta(a^1 \otimes_B a^2)$.

Endomorphism Ring Theorem. If $A \,|\, B$ is right D2, then $A^{\text{op}} \xrightarrow{\rho} \text{End}\,_B A$ is a left S-Galois extension, where $S = \text{End}\,_B A_B$.

Explanation. S is a left $R = C_A(B)$-bialgebroid: very briefly, $\Delta_S(\alpha)(a \otimes_B a') = \alpha(aa')$ and $\varepsilon_S(\alpha) = \alpha(1) \in R$. R-dual to T: $\langle \alpha, t \rangle = \alpha(t^1)t^2$ nondegenerate pairing.

$\mathcal{E} = \text{End}\,_B A$, left S-comodule algebra w/ coact. ${}^S\rho : \mathcal{E} \to S \otimes_R \mathcal{E}$, $f_{(-1)} \otimes f_{(0)} = \sum_j \gamma_j \otimes_R f \triangleleft u_j$ where $f \triangleleft t = t^1 f(t^2 -)$ is a right T-action on $\mathcal{E}$.

Coinvariants ${}^{\text{co}\,\text{S}}\mathcal{E} = \rho(A)$;
e.g. $\sum_j \gamma_j \otimes_R u_j^1 u_j^2(?a) = A \otimes \rho_a$.

Finally $S \otimes_R \mathcal{E}$ is a Galois coring:

$$\mathcal{E} \otimes_{\rho(A)} \mathcal{E} \xrightarrow{\ \beta\ } S \otimes_R \mathcal{E}$$

$$\mathcal{E} \otimes_{\rho(A)} \mathcal{E} \xrightarrow{\cong} \mathcal{E} \otimes_A \mathcal{E} \xrightarrow{\cong} \mathrm{Hom}\,({}_B\mathrm{Hom}\,(\mathcal{E}_A, A_A), {}_BA) \xrightarrow{\cong} \mathrm{Hom}\,({}_BA \otimes_B A, {}_BA) \xrightarrow{\cong} S \otimes_R \mathcal{E}$$

Same idea applied to Frobenius bimodule ${}_RM_S$ (i.e. ${}_RM$ and M_S are finite projective and have S-R-isomorphic duals):

Definition. A Frobenius bimodule ${}_RM_S$ is D2 if both $R \xrightarrow{\lambda} \mathrm{End}\, M_S$ and $S \xrightarrow{\rho} (\mathrm{End}\, {}_RM)^{\mathrm{op}}$ are D2 extensions.

For example, $g : B \to A$ is D2 Frob. ext. $\Leftrightarrow$ ${}_BA_A$ or ${}_AA_B$ is D2 bimodule, since $\lambda : B \to \mathrm{End}\, A_A \cong A$ is same as $g : B \to A$ and $A \to (\mathrm{End}\, {}_BA)^{\mathrm{op}}$ is D2 if $A \,|\, B$ is D2.

This definition is consistent with Szlachanyi's right depth two arrow in a modified bicategory of bimodules [S].

Some questions to part with:

1. Is there a left D2 algebra extension that is not right D2?

2. Is a D2 Hopf subalgebra normal?

3. same question for semisimple Hopf subalgebra: can apply theory of D2 Frobenius ext. of Böhm-Szlachanyi.

4. A bicategory of corings (Brzezinski et al. [BKG]) may imply a D2 coring: is there a viable theory in this?

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