Introduction to Depth Two and a Galois Theory for Bialgebroids

Lars Kadison

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Origins of depth two and finite depth in C^*- and von Neumann algebras

Bratteli diagram for inclusion of fin. dim. C^*-algebras: e.g. $S_2 < S_3$ and their corresp. group $\mathbb{C}$-algebras is unitarily equiv. to inclusion mapping $(\lambda, \mu) \mapsto (\lambda, \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \mu)$

Inclusion matrix is induction-restriction table for irreducible characters:

$S_2 \leq S_3$	χ_1	χ_2	χ_3
ψ_1	1	1	0
ψ_2	0	1	1

Jones tower of von Neumann factors:

$$N \subset M \subset M_1 = \mathrm{End}\, M_N \subset M_2 \subset \cdots$$

(iteration by basic End construction). Basic construction of f.d. C^*-algebras has a mirror image Bratteli diagram.

Derived tower of centralizers or relative commutants:

$$C_N(N) \subset C_M(N) \subset C_{M_1}(N) \subset C_{M_2}(N) \subset \cdots$$

Draw Bratteli diagrams of each inclusion : $N \subset M$ is depth n if at n'th level it begins to repeat itself via reflections.

The notion depth two (D2) extends to any ring extension (Nikshych-L.K., Szlachanyi-L.K.)

Let $B \subseteq A$ be a subring with $1_B = 1_A$ or $B \to A$ a unital ring homomorphism. $A \,|\, B$ is D2 if $A \otimes_B A \in Add(A)$ as natural A-B-bimodules (right D2) and B-A-bimodules (left D2).

Dress equiv. of cat's: $Add(M_C) \cong FGP_{\mathrm{End}\, M_C}$ for any module M over ring C,

$Add(A) \cong FGP_R$ where $R \cong \mathrm{End}\,{}_BA_A$ is centralizer $C_A(B)$.

Since Hom $(A, A \otimes_B A) \cong (A \otimes_B A)^B := T$
and Hom $(A \otimes_B A, A) \cong \mathrm{End}\,{}_BA_B := S$
we have equiv. def. of D2 extension:

(left D2 quasibase) $\exists \beta_i \in S, t_i \in T$:

$$a \otimes a' = \sum_{i=1}^{n} t_i \beta_i(a) a'$$

(right D2 quasibase) $\exists \gamma_j \in S, u_j \in T$:

$$a \otimes a' = \sum_{j=1}^{m} a \gamma_j(a') u_j$$

Note: T_R and ${}_RS$ are f.g. projective by Dress equivalence or the left D2 eq. above. Also $Add(A) = Add(A \otimes_B A)$ and indeed obtain via Hirata that R and $(\mathrm{End}\, A \otimes_B A_A)^B$ are Morita equivalent.

Example. A f.g. projective algebra A. If $x_i \in A$ and $p_i \in A^*$ satisfies $\mathrm{id}_A = \sum_i^n x_i p_i$, then $a \otimes a' = \sum_{i=1}^n a p_i(a') 1 \otimes x_i$.

Example. A normal subgroup of a finite group: $N \triangleleft G$. If $G = \coprod_{i=1}^n g_i N$, $A = \mathbb{C}\, G$ and $B = \mathbb{C}\, N$, then $t_i = g_i \otimes g_i^{-1}$ and $\beta_i(\sum_{g \in G} a_g g) = \sum_{g \in g_i N} a_g g$ is a left D2 quasibase since for all $g \in G$

$$g \otimes_N 1 = \sum_{i=1}^n g_i \otimes g_i^{-1} \beta_i(g)$$

Example. Hopf-Galois Ext's.
H = bialgebra of $\dim H = n$, A = an H-comodule algebra w/ coaction $A \to A \otimes H$ s.t. $B = A^{\mathrm{co\,H}}$ and *i*somorphism

$$\beta : A \otimes_B A \to A \otimes H, \quad \beta(a \otimes a') = a{a'}_{(0)} \otimes {a'}_{(1)}$$

Then ${}_A A \otimes_B A_B \cong \oplus^n {}_A A_B$.
H has an antipode [Sch], so
$\beta'(a \otimes a') = a_{(0)} a' \otimes a_{(1)}$ is isomorphism, and $A|B$ is left D2 as well.
E.g. normal Hopf subalg's are D2.

$C_A(B)$ IS NORMAL SUBALG [R, LK5]

Given $A\,|\,B$ D2, $\forall$ ideal $I \subset A$

$$(I \cap R)A = A(I \cap R) \qquad (1)$$

Reason: $\forall r \in R \cap I$, $\exists A \otimes_B A \to I$, $x \otimes_B y \mapsto xry$, apply to $1 \otimes_B a = \sum_j \gamma_j(a) u^1_j \otimes_B u^2_j$ to get

$$ra = \sum_j \gamma_j(a) u^1_j r u^2_j \in A(I \cap R)$$

Similarly $ar = \sum_i t^1_i r t^2_i \beta_i(a) \in (I \cap R)A$, whence eq. (1).

OBS. Using the similar equations

$$a_{(1)} \otimes \tau(a_{(2)}) a_{(3)} = a \otimes 1$$

$$a_{(1)} \tau(a_{(2)}) \otimes a_{(3)} = 1 \otimes a$$

for $a \in$ a Hopf algebra H w/ antipode τ, a normal Hopf subalgebra $K \subseteq H$ (i.e. every $x \in H$ satisfying $\tau(x_{(1)}) K x_{(2)} \subseteq K$ and $x_{(1)} K \tau(x_{(2)}) \subseteq K$) is a normal subring. Converse follows from $HK^+ = K^+ H$ characterization (where $K^+ = \ker \varepsilon \cap K$).

THEOREM (Sz.-L.K.) Given left or right D2 ext $A\,|\,B$, the ring $S = \mathrm{End}\,{}_B A_B$ is a (f.g. projective left) bialgebroid over $R = C_A(B)$ and $T = (A \otimes_B A)^B$ is its R-dual (f.g. proj. right) bialgebroid.

Right R-bialgebroid structure on T:

1. Note $T := (A \otimes_B A)^B \overset{\cong}{\longleftarrow} \mathrm{End}\,({}_A A \otimes_B A_A)$ via $F \mapsto F(1 \otimes 1)$, which induces the multiplication on T:
$$tu = u^1 t^1 \otimes_B t^2 u^2, \quad 1_T = 1_A \otimes 1_A$$
2. dual groupoid set-up: $R \xrightarrow{s_R} T \xleftarrow{t_R} R^{\mathrm{op}}$ via $s_R(r') = 1 \otimes_B r'$ and $t_R(r) = r \otimes_B 1$ satisfying $s_R(r')t_R(r) = t_R(r)s_R(r')$.
3. bimodule ${}_R T_R = T_{s_R, t_R}$: $r \cdot t \cdot r' = rt^1 \otimes t^2 r'$.

4. (T, Δ, ε) is R-coring where comultiplication $\Delta_T : T \to T \otimes_R T \cong (A \otimes_B A \otimes_B A)^B$, $\Delta(t) = t_{(1)} \otimes t_{(2)} = t^1 \otimes_B 1 \otimes_B t^2$ and counit $\varepsilon_T(t) = t^1 t^2 \in R$.

5. bialgebroid properties: $\Delta(1_T) = 1_T \otimes_R 1_T$, $\Delta(tu) = \Delta(t)\Delta(u)$, $\varepsilon(1_T) = 1_R$, and right properties $s_R(r)u_{(1)} \otimes u_{(2)} = u_{(1)} \otimes t_R(r)u_{(2)}$, $\varepsilon(tu) = \varepsilon(t_R(\varepsilon(t))u) = \varepsilon(s_R(\varepsilon(t))u)$.

Left R-bialgebroid structure on $S = \mathrm{End}\,{}_B A_B$: briefly, left and right mult. $R \xrightarrow{\lambda} S \xleftarrow{\rho} R^{\mathrm{op}}$, coproduct is dual of mult.

$$\Delta_S(\alpha)(a \otimes_B a') = \alpha(aa')$$

Formula works because of cup product result for relative Hochschild cochains of D2 ext., $C^2(A, B; A) \cong C^1(A, B; A) \cup C^1(A, B; A)$. [LK4] Counit $\varepsilon_S(\alpha) = \alpha(1) \in R$.

S is R-dual to T: pairing
$\langle \alpha, t \rangle = \alpha(t^1)t^2 \in R$ nondegenerate if left D2.

Theorem [NK, SK, LK] An algebra extension $A\,|\,B$ is a right T-Galois ext. for some left fin. projective right R-bialgebroid T $\Leftrightarrow$ $A\,|\,B$ is right D2 and balanced.

Explanation. $\Rightarrow$ is rather like the example above. ($\Leftarrow$) A is a right T-comodule algebra: coaction $\delta : A \to A \otimes_R T$, $a_{(0)} \otimes_R a_{(1)} = \sum_j \gamma_j(a) \otimes_R u_j$ satisfies $\delta(1_A) = 1_A \otimes_R 1_T$ and $\delta(xy) = \delta(x)\delta(y)$.

Coinvariants $A^{\mathrm{co}\,\mathrm{T}} = \{x \in A |\, x_{(0)} \otimes_R x_{(1)} = x \otimes 1\} = B$ where $\subseteq$ follows from A_B balanced.

Finally $A \otimes_R T$ is Galois A-coring [BW] with bimodule $a(a' \otimes t)a'' = aa'a''_{(0)} \otimes_R ta''_{(1)}$, comult. $A \otimes \Delta_T$ and counit $A \otimes \varepsilon_T$. Grouplike elt. $1_A \otimes 1_T$ and isomorphism $A \otimes_R T \cong A \otimes_B A$ via $a \otimes_R t \mapsto at^1 \otimes_B t^2$ w/ inverse

$$a \otimes_B a' \longmapsto \sum_j a\gamma_j(a') \otimes_R u_j = aa'_{(0)} \otimes a'_{(1)}$$

the Galois isomorphism. $\square$

Different Hopf algebroids that turn up via depth two theory:

D2 Ext. $A \,|\, B$	*Bialgebroid* End $_B A_B$ *or Dual* $(A \otimes_B A)^B$
Fin. proj. alg. A [Lu]	End A and A^e
Irred. Frob. ext. [KN1]	dual Hopf algs.
Frob. ext. w/ sep. cent. [KS]	dual weak Hopf algs. [KN2]
H-separable ext. [LK0]	Hopf algebroid R^e
Hopf-Galois ext. [LK1]	Lu Geometric Hopf algebroid
Pseudo-Galois ext. [LK2]	Connes-Mosc. Hopf algebroi

Example. An H-extension $A \,|\, B$ w/ split injective Galois (A-B-)mapping $\beta : A \otimes_B A \hookrightarrow A \otimes H$ is D2, therefore T-Galois.
If β is $\cong$, then $T^{\mathrm{op}} \cong R \# H^{\mathrm{op}}$ is a Lu geometric Hopf algebroid where H acts by $r \triangleleft h = a^1 r a^2$ and $1 \otimes h = \beta(a^1 \otimes_B a^2)$.

Anchor maps

Ping Xu quantizes certain Lie algebroids like tangent bundle TX over Poisson manifold X

to obtain noncocommutative Hopf algebroids such as twisted differential operators $\mathcal{D}_q(X)$. A Lie algebroid is a vector bundle over X equipped with a Lie bracket and a Lie homomorphism and bundle map into TX called anchor map.

For a Hopf algebroid H over R, the anchor map μ is the unit representation $H \to \operatorname{End} R$ in the tensor category of H-modules.

Interesting anchor maps in depth two theory:

$\mu : S \to \operatorname{End} R$ is evaluation of $\operatorname{End}\, {}_B A_B$ on the centralizer R.

$\mu : T \to \operatorname{End} R$ given by $r \triangleleft t = t^1 r t^2$, a generalized Miyashta-Ulbrich action of Hopf algebra on centralizer of Hopf-Galois extension, or quotient group conjugation action on centralizer of normal subgroup [KK].

By comparing anchors of Hopf algebroid in [LK2] and in [CM] you may write down an isomorphism between these two objects.

Some questions and problems related to depth two:

1. Chirality Problem.
   Is there a left D2 algebra extension that is not right D2?

2. Normality Problem.
   Is a D2 Hopf subalgebra normal?

3. Galois Correspondence Problem.
   Find new conditions under which D2 subextensions correspond to Hopf subalgebroids.

4. Inverse Galois Problem.
   Given a f.g. projective left bialgebroid, is it the endomorphism ring of a D2 extension?

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