

Finite Depth and a Galois Correspondence

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Similarity of Functors $F \sim G$

Two functors between abelian categories $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are said to be *similar* if \forall object $X \in \mathcal{C}$, $\exists n, m \in \mathbb{Z}_+$: two naturally split epis

$$F(X)^n \xrightarrow{\dot{\rightarrow}_{\oplus}} G(X) \rightarrow 0, \quad G(X)^m \xrightarrow{\dot{\rightarrow}_{\oplus}} F(X) \rightarrow 0$$

i.e. $G(X) \oplus * \cong F(X)^n : G | F$

$$F(X) \oplus * \cong G(X)^m : F | G$$

Easy Consequence i.c.o. module categories:
endomorphism rings $\text{End } F(X)$ and $\text{End } G(X)$
are Morita equivalent.

Definition. Given adjoint pair of functors (F, G) between module categories

$$B - \mathbf{Mod} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} A - \mathbf{Mod}$$

F is rD2 ("right depth two") if $FGF \sim F$.

$$F = M \otimes_B -, \quad G = \text{Hom}({}_A M, {}_A -), \quad E := \text{End } M_B$$

Def 2. The functor F is rD3 if

$$\text{Ind}_A^E F \sim \text{Ind}_A^E \text{Res}_A^E \text{Ind}_A^E F$$

Jones, "Iterate endo-ring construction to obtain E_2, E_3, \dots " (with $E_1 = E, E_0 = A, E_{-1} = B$)

The functor F is rD_n if

$$\text{Ind}_A^{E_{n-2}} F \sim \text{Ind}_{E_{n-3}}^{E_{n-2}} \text{Res}_{E_{n-3}}^{E_{n-2}} \text{Ind}_A^{E_{n-2}} F$$

Assume QF-functor where $({}_B M_A)^* \sim {}^*({}_B M_A)$ and finite projectivity:

Thm. If F is D_n , then $\text{Ind}_A^{E_m} F$ is D_2 for some $m \geq n - 2$.

Rest of talk: ring homom. $B \rightarrow A \rightsquigarrow$
 bimodule $M = {}_A A_B$, $F(-) = A \otimes_B - = \text{Ind}_B^A$,
 $G(-) = \text{Hom}({}_A A, {}_A -) = \text{Res}_B^A$.

Def. "Ring extension" $B \rightarrow A$ is rD2 if
 $A \otimes_B A \sim A$ as natural A - B -bimodules:
 $A \oplus * \cong A \otimes_B A$ always the case,
 Defining condition for rD2:

$$1) {}_A A \otimes_B A_B \oplus * \cong {}_A A^n_B$$

Other equivalent conditions for rD2 extension
 $A|B$:

2) $\exists \gamma_i \in \text{End}_B A_B$, $u_i \in (A \otimes_B A)^B$: within
 $A \otimes_B A$,

$$1 \otimes a = \sum_i \gamma_i(a) u_i^1 \otimes_B u_i^2 \text{ (called right D2 qb)}$$

3) $\text{End}_B A \sim A$ as B - A -bimodules, if ${}_B A$ is finite
 projective

4) $\text{CoInd}_B^A \sim \text{CoInd}_B^A \text{Res}_B^A \text{CoInd}_B^A$
 (where $\text{CoInd}_B^A V = \text{Hom}(A_B, V_B)$) if A_B is fin. proj.

5) $\text{Res}_B^A \sim \text{Res}_B^A \text{CoInd}_B^A \text{Res}_B^A$

PF: $\text{CoInd} \sim \text{CoIndResCoInd} \xrightarrow{B^A \text{ f.p.}}$
 $A \otimes_B A \oplus * \cong A^n$

Coinduction $\text{Hom}(A_B, V_B) \cong V \otimes_B A^*$ by fin. proj. of A_B

So hypothesis yields $A^* \otimes_B A^* \sim A^*$ as B - A -bimod's.

Note $A^* \otimes_B A^* \cong \text{Hom}(A \otimes_B A_B, B_B)$
 via $\alpha \otimes \beta \mapsto (x \otimes y \mapsto \alpha(\beta(x)y))$

Apply $\text{Hom}({}_B -, {}_B B)$ and reflexivity of modules, obtain $A \otimes_B A \sim A$ as A - B -bimod's, i.e. D2'ity of $B \rightarrow A$.

Examples of D2 Ring Extensions

1. Any automorphism $A \xrightarrow{\alpha} A$.
2. A normal subgroup $N \triangleleft G$ or normal Hopf subalgebra $K \subseteq H$ (where $HK^+ = K^+H$).
3. Hopf- H -Galois extension $A|B$, "quantum principal bundles"

coaction $A \rightarrow A \otimes H$, $A = H$ -comodule algebra.
 $B =$ coinvariants, $b_{(0)} \otimes b_{(1)} = b \otimes 1$

Galois mapping $\beta : A \otimes_B A \rightarrow A \otimes H$,
 $\beta(x \otimes y) = xy_{(0)} \otimes y_{(1)}$ is \cong , so $A \otimes_B A \cong A^{(I)}$
as A - B -bimodules: $I =$ basis for H .

4. "Pseudo-Hopf-Galois Extensions" - where β above is split injection of bimodules.
5. H -separable extensions - satisfying a stronger condition than D2'ity: ${}_A A \otimes_B A_A \oplus * \cong {}_A A^n A$.

Theorems for an rD2 extension $A|B$:

1. $\text{End } {}_B A_B = S$ is a left *bialgebroid* over centralizer $A^B = R$.

source, $\lambda : R \rightarrow S$, left mult.; target $\rho : R \rightarrow S$, right mult.

ring, $\alpha \circ \beta$, $\alpha + \beta$, $1 = \text{id}_A$.

coring, $\Delta : S \rightarrow S \otimes_R S \cong \text{Hom}_{B-B}(A \otimes_B A, A)$
 $\Delta(\alpha)(x \otimes y) = \alpha(xy)$, $\varepsilon(\alpha) = \alpha(1_A)$.

2. Action $S \otimes_R A \rightarrow A$, $\alpha \triangleright a = \alpha(a)$, $A = S$ -module algebra w/ invariant subalg.

$A^S \cong \text{End}_E A$ where $E = \text{End } A_B$.

3. Ring isomorphism $\text{End } A_B \cong A \rtimes S$, crossed product, via $a \otimes \alpha \mapsto \lambda_a \circ \alpha$.

4. Dual theory: $(A \otimes_B A)^B = T$ right R -bialgebroid w/ R -pairings $S \otimes_R T \rightarrow R$,
 $\langle \alpha, t \rangle = \alpha(t^1)t^2$ and $[u, \beta] = u^1\beta(u^2)$.

5. Suppose $A^B \subseteq B$, then S becomes Hopf algebroid w/ antipode τ given by

$$\alpha(t^1)t^2 = t^1\alpha^\tau(t^2)$$

Galois D2 Theory for Coactions

Coaction $A \rightarrow A \otimes_R T$, $a \mapsto \sum_i \gamma_i(a) \otimes u_i$, using right D2 qb $\gamma_i \in \text{End}_B A_B$, $u_i \in (A \otimes_B A)^B$

$A = T$ -comodule algebra

$B = A^{\text{co}T}$ if A_B is faithfully flat or balanced

Galois isomorphism $\beta : A \otimes_B A \rightarrow A \otimes_R T$,
 $x \otimes y \mapsto \sum_i x \gamma_i(y) \otimes u_i$, inverse: $a \otimes_R t \mapsto at^1 \otimes t^2$

Main Theorem: Ring extension $A|B$ is Galois wrt. right bialgebroid T which is left projective over a base ring $R \Leftrightarrow A|B$ is rD2 and balanced.

Analogous to "Galois \Leftrightarrow normal + separable" in field theory.

Galois Correspondence

- generalize D2 extension to D3 tower

rD3 tower of rings $A \supseteq B \supseteq C$ if

$${}_A A \otimes_B A_C \oplus * \cong {}_A A^n C.$$

Suppose $A \supseteq B \supseteq C$ enjoys a Jacobson-Bourbaki correspondence with $\text{End } A_C$ and its "closed" subrings \mathcal{R} . (E.g. $A =$ central simple algebra, $B, C =$ fields, or all are division rings. One way to do classical Galois theory.)

I.e. $B \rightsquigarrow \text{End } A_B \subseteq \text{End } A_C$, and
inverse $\mathcal{R} \rightsquigarrow \text{End } \mathcal{R}A$.

Suppose moreover $A|C$ is D2. Then there is Galois correspondence between D3 intermediate ring B and "closed" left coideal subring $\text{End } {}_C A_B$ of A^C -bialgebroid $\text{End } {}_C A_C$. The mappings are given by $B \rightsquigarrow \text{End } B A_C$ with inverse $\mathcal{J} \rightsquigarrow A^{\mathcal{J}}$.

Some parting thoughts and questions

D3 tower of groups $G > H > K$ if normal closure $K^G < H$ (over any field k , form group algebras).

What is D3 subgroup $G > H$?

(Where $E | A | B$ is D3 tower, $B = k[H]$,
 $A = k[G]$, $E = \text{End } A_B$)

What is depth $n \geq 3$ subgroup $G > H$?

(Where $E_{n-2} | E_{n-3} | B$ is D3 tower)

Believe that if $[G : H] < \infty$ and $\text{char} = 0$ answer is just all subgroups, since $B \subseteq E$ is H-separable and $B \subseteq A$ is separable extension. More interesting in modular representations? quantum groups?

Bibliography to talk is available through my homepage.