Depth two and Hopf algebroids

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Given a normal subgroup $N \triangleleft G$, we obtain its quotient group G/N.

More generally, given a normal Hopf subalgebra $K \triangleleft H$ (invariant under the left and right adjoint actions) there is the quotient Hopf algebra H/HK^+.

But for a unital subalgebra $B \subseteq A$ there has not been a notion of normality with quotient object until recently. We claim it is "depth two subalgebra" or "depth two extension $A \,|\, B$" with its quotient object being the bialgebroid, at times Hopf algebroid, $\mathrm{End}\, {}_B A_B$ (the algebra of bimodule endomorphisms) and its dual $(A \otimes_B A)^B \cong \mathrm{End}\, {}_A A \otimes_B A_A$ (the B-central tensors).

A Hopf algebroid is roughly speaking a Hopf algebra over a noncommutative base algebra, which in case of depth two extension $A \,|\, B$ is the centralizer,

$$C_A(B) = \{r \in A \,:\, \forall\, b \in B, rb = br\}.$$

There is of course a price to pay for this generalization of normality and quotient object: e.g., in our first example, G/N is replaced by an algebra containing the semi-direct product $C_G(N) \rtimes G/N$ in its group of units, where G/N acts on the centralizer $C_G(N)$ by the Miyashta action $x \triangleleft gN = g^{-1}xg$.

The rest of the talk addresses:

1. What is a Hopf algebroid?
2. What examples are there?
3. What is depth two for subrings or algebra extensions?
4. What examples are there?
5. How do we obtain bialgebroids from depth two extensions?
6. What are their Galois actions?

DEFINITION OF HOPF ALGEBROID with total algebra S and base algebra R [Lu, BoSz]:

1. S and R are (unital associative) algebras,
2. there are commuting algebra homomorphisms $R \xrightarrow{\lambda} S \xleftarrow{\rho} R^{\mathrm{op}}$ (called *source* and *target*, respectively) where for each $r, s \in R$ we have $\lambda(r)\rho(s) = \rho(s)\lambda(r)$,
3. an R-R-bimodule ${}_R S_R$ given by
$$r \cdot \alpha \cdot s = \lambda(r)\rho(s)\alpha \quad (\alpha \in S)$$
4. S is an R-*coring* with *comultiplication* $\Delta : S \to S \otimes_R S$ given by notation $\Delta(\alpha) = \alpha_{(1)} \otimes \alpha_{(2)}$ and *counit* $\varepsilon : S \to R$, where Δ is coassociative and for all $\alpha \in S$,
$$\varepsilon(\alpha_{(1)}) \cdot \alpha_{(2)} = \lambda(\varepsilon(\alpha_{(1)}))\alpha_{(2)} = \alpha$$

$$\alpha_{(1)} \cdot \varepsilon(\alpha_{(2)}) = \rho(\varepsilon(\alpha_{(2)}))\alpha_{(1)} = \alpha$$

5. unitality conditions: $\varepsilon(1_S) = 1_R$ and $\Delta(1_S) = 1_S \otimes_R 1_S$.

6. multiplicativity of Δ with technical pre-condition: for all $\alpha, \beta \in S$, $r \in R$,

$$\alpha_{(1)}\rho(r) \otimes_R \alpha_{(2)} = \alpha_{(1)} \otimes_R \alpha_{(2)}\lambda(r)$$

$$\Delta(\alpha\beta) = \Delta(\alpha)\Delta(\beta)$$

7. modified multiplicativity of ε:

$$\varepsilon(\alpha\beta) = \varepsilon(\alpha\lambda(\varepsilon(\beta))) = \varepsilon(\alpha\rho(\varepsilon(\beta)))$$

The axioms above define a (left) R-bialgebroid S. In addition, a Hopf algebroid (with bijective antipode) satisfies:

8. there is an anti-automorphism $\tau : S \to S$ such that

9. $\tau \circ \rho = \lambda$,

10. $\tau^{-1}(\alpha_{(2)})_{(1)} \otimes_R \tau^{-1}(\alpha_{(2)})_{(2)}\alpha_{(1)} = \tau^{-1}(\alpha) \otimes_R 1_S$,

11. $\tau(\alpha_{(1)})_{(1)}\alpha_{(2)} \otimes_R \tau(\alpha_{(1)})_{(2)} = 1_S \otimes_R \tau(\alpha)$ $(\forall\, \alpha \in S)$

The last two are axioms in [BoSz] that replace an unwieldy axiom in [Lu], although not entirely equivalent.

Example 1. Lu's ("coarse") Hopf algebroid $A^e := A \otimes A^{\mathrm{op}}$ over an algebra A (f.g. projective over a commutative ground ring). Its structure:

Multiplication: $(a \otimes b)(c \otimes d) = ac \otimes db$,

Identity: $1_A \otimes 1_A$,

Source: $\lambda(a) = a \otimes 1$,

Target: $\rho(a) = 1 \otimes a$,

A-A-Bimodule: $a \cdot (b \otimes c) \cdot d = ab \otimes cd$,

Comultiplication: $\Delta(a \otimes b) = (a \otimes 1) \otimes_A (1 \otimes b)$,

Counit: $\varepsilon(a \otimes b) = ab$

Antipode: $\tau(a \otimes b) = b \otimes a$, $\forall\, a, b \in A$.

Example 2. Lu's bialgebroid $\operatorname{End} A$ of linear endomorphisms over an algebra A.

Multiplication: $f \circ g$, composition,

Identity: id_A,

Source: $\lambda(a) =$ left multiplication by $a \in A$,

Target: $\rho(a) =$ right multiplication by $a \in A$,

A-A-Bimodule: $a \cdot f \cdot b = af(-)b$,

Comultiplication: $\Delta(f)(a \otimes b) = f(ab)$, since $\operatorname{End} A \otimes_A \operatorname{End} A \cong \operatorname{Hom}(A \otimes A, A)$ via
$f \otimes g \longmapsto (a \otimes b \mapsto f(a)g(b))$,

Counit: $\varepsilon(f) = f(1)$

Antipode: only in special cases such as Azumaya algebra A.

Example 3. Lu's Poisson Geometric Hopf Algebroid. Given: H = Hopf algebra with antipode τ_H,
$D(H)$ = the Drinfeld double of H (= $H^* \otimes H$ linear structure),
V = a left $D(H)$-module algebra such that the R-matrix $\sum_i 1 \otimes h_i \otimes p_i \otimes 1$ (where $h_i \in H$, $p_i \in H^*$ are dual bases) satisfies for every $u, v \in V$:

$$vu = \sum_i (p_i \cdot u)(h_i \cdot v)$$

Then the crossed product $V \rtimes H$ is a Hopf algebroid over V, with
Multiplication:
$(v \rtimes h)(w \rtimes k) = v(h_{(1)} \cdot w) \rtimes h_{(2)}k$,
Identity: $1_V \rtimes 1_H$,
Source: $\lambda(v) = v \rtimes 1_H$,
Target: $\rho(v) = \sum_i (p_i \cdot v) \rtimes h_i$,
Comultiplication:
$\Delta(v \rtimes h) = v \rtimes h_{(1)} \otimes_V 1_V \rtimes h_{(2)}$,
Counit: $\varepsilon(v \rtimes h) = \varepsilon_H(h)v$,
Antipode:
$\tau(v \rtimes h) = \sum_i (1 \rtimes \tau_H(h))\rho(\tau_H^2(h_i) \cdot p_i \cdot v)$.

Example 4. The Connes-Moscovici Hopf Algebroid $A \odot H \odot A$ where H is an involutive Hopf algebra and A is left H-module algebra [CM].

Linear space: $A \odot H \odot A = A \otimes H \otimes A$

Multiplication:
$(a \odot h \odot b)(c \odot k \odot d) = a(h_{(1)} \cdot c) \odot h_{(2)} k \odot (h_{(3)} \cdot d) b$

Identity: $1_A \odot 1_H \odot 1_A$

Source: $\lambda(a) = a \odot 1_H \odot 1_A$

Target: $\rho(a) = 1_A \odot 1_H \odot a$

Comultiplication:
$\Delta(a \odot h \odot b) = (a \odot h_{(1)} \odot 1_A) \otimes_A (1_A \odot h_{(2)} \odot b)$

Counit: $\varepsilon(a \odot h \odot b) = a \varepsilon_H(h) b$

Antipode [KR]:
$\tau(a \odot h \odot b) = \tau_H(h_{(3)}) \cdot b \otimes \tau_H(h_{(2)}) \otimes \tau_H(h_{(1)}) \cdot a$

Depth Two [KS]: A Normality Property for Subalgebras

Let $B \subseteq A$ be a subalgebra.

The subalgebra is depth two (or D2) if there is a positive integer N such that

$$A \otimes_B A \oplus * \cong A^N$$

1. as B-A-bimodules (left D2), and

2. as A-B-bimodules (right D2).

This definition also works for an algebra homomorphism $B \to A$ or a ring extension $A \,|\, B$.

Theorem (cf. [KS]). Either of conditions 1 or 2 imply End ${}_B A_B$ is a bialgebroid over the centralizer $R := C_A(B)$. It has R-dual “right” bialgebroid

$$(A \otimes_B A)^B = \{t \in A \otimes_B A \, : \, \forall\, b \in B,\, bt^1 \otimes t^2 = t^1 \otimes t^2 b\}$$

D2 Ext. $A \mid B$	*Bialgebroid* $\mathrm{End}\,{}_BA_B$ *or Dual* $(A \otimes_B A)^B$
Fin. proj. alg. A	Lu's $\mathrm{End}\,A$ and A^e
H-separable ext.	Coarse Hopf algebroid
Hopf-Galois ext.	Lu's Geometric Hopf algebroid
Pseudo-Galois ext.	Connes-Moscovici Hopf algebroid

The bialgebroid $\text{End}\,{}_BA_B$ over the centralizer of a D2 extension $A\,|\,B$ is a generalization of Lu's linear endomorphism bialgebroid over an algebra.
Multiplication on $\text{End}\,{}_BA_B$ is composition.
Identity: id_A,
Source: $\lambda(r) =$ left multiplication by $r \in C_A(B) = \{r \in A\, :\, \forall\, b \in B,\, br = rb\}$,
Target: $\rho(r) =$ right multiplication by $r \in C_A(B)$,
Comultiplication: $\Delta(\alpha)(x \otimes_B y) = \alpha(xy)$
using an isomorphism
$\text{End}\,{}_BA_B \otimes_R \text{End}\,{}_BA_B \cong \text{Hom}\,({}_BA \otimes_B A_B, {}_BA_B)$
via $\alpha \otimes_R \beta \longmapsto (x \otimes_B y \mapsto \alpha(x)\beta(y))$,
Counit: $\varepsilon(\alpha) = \alpha(1)$,
Antipode: only in special cases, e.g. H-separable extension $A\,|\,B$, which satisfies $A \otimes_B A \oplus * \cong A^N$ as A-A-bimodules. Then $\text{End}\,{}_BA_B \cong R \otimes_Z R^{\text{op}}$ (where $R = C_A(B)$ is f.g. projective over $Z =$ the center of A) via $r \otimes s \mapsto \lambda(r) \circ \rho(s)$, Lu's coarse Hopf algebroid.

Duality with $(A \otimes_B A)^B$ given by nondegenerate pairing, $(\forall\, t \in (A \otimes_B A)B, \alpha \in \text{End}\,_B A_B)$

$$\langle \alpha,\, t \rangle = \alpha(t^1)t^2 \in C_A(B)$$

Multiplication on $(A \otimes_B A)^B$:
$tu = (t^1 \otimes_B t^2)(u^1 \otimes_B u^2) = u^1 t^1 \otimes_B t^2 u^2$.

Identity: $1_A \otimes_B 1_A$,

Source: $\lambda(r) = 1_A \otimes r$,

Target: $\rho(r) = r \otimes 1_A$, $(\forall\, r \in R = C_A(B))$

R-R-Bimodule: $r \cdot t \cdot s = rt^1 \otimes_B t^2 s$,

Comultiplication: $\Delta(t) = t^1 \otimes_B 1_A \otimes_B t^2$, using $(A \otimes_B A)^B \otimes_R (A \otimes_B A)^B \cong (A \otimes_B A \otimes_B A)^B$ via $t \otimes_R u \mapsto t^1 \otimes_B t^2 u^1 \otimes_B u^2$.

Counit: $\varepsilon(t) = t^1 t^1$.

Antipode: only in special cases, e.g. $A \mid B$ is a Hopf-Galois extension, which has Galois map $\beta : A \otimes_B A \xrightarrow{\cong} A \otimes H$ where H is a finite dimensional Hopf algebra. Then $(A \otimes_B A)^B \cong R \otimes H$, and multiplication on LHS transfers to the smash product multiplication,

$$(r \otimes h)(s \otimes k) = s(r \triangleleft k_{(1)}) \otimes hk_{(2)}$$

$(r, s \in R = C_A(B), h, k \in H)$ where right action of H on R is Miyashta-Ulbrich's action $r \triangleleft h = t^1 r t^2$ where $\beta(t^1 \otimes_B t^2) = 1 \otimes h$. The multiplication above is just the opposite of Lu's geometric Hopf algebroid $R \rtimes H^{\mathrm{op}}$.

Example 4 of D2 extension: a pseudo-Galois extension $A \,|\, B$ [MM], which satisfies (for some pos. integer N)

$$A \otimes_B A \oplus * \cong \oplus_{\sigma \in G} A_\sigma^N$$

where G is a finite subgroup of algebra automorphisms of A fixing each element of B, and A_σ denotes the twisted A-module given by $a \cdot x = a\sigma(x)$ for $\sigma \in G$.

Then End ${}_B A_B$ is linearly generated by endomorphisms $\lambda(r) \circ \sigma \circ \rho(s)$ where $r, s \in R = C_A(B)$ and $\sigma \in G$. Note that for $r, s, u, v \in R$ and $\sigma, \tau \in G$:

$$\lambda(r)\sigma\rho(s) \circ \lambda(u)\tau\rho(v) = \lambda(r\sigma(u)) \circ \sigma\tau \circ \rho(v\tau^{-1}(s))$$

Thus there is an R-bialgebroid epimorphism $R^e \bowtie G \to$ End ${}_B A_B$ given by

$$r \otimes s \bowtie \sigma \longmapsto \lambda(r) \circ \sigma \circ \rho(s)$$

where multiplication on $R^e \bowtie G$ is given by

$$(r \otimes s \bowtie \sigma)(u \otimes v \bowtie \tau) = r\sigma(u) \otimes v\tau^{-1}(s) \bowtie \sigma\tau$$

Via the depth two machinery above it is not hard to write down a Hopf algebroid structure more generally on $A^e \bowtie H$ where H is an involutive Hopf algebra with left H-module algebra A [LK2].

The Hopf algebroid $A^e \bowtie H$ where H is Hopf algebra with antipode satisfying $\tau_H^2 = \mathrm{id}_H$ and A is left H-module algebra:

Multiplication: $(a \otimes b \bowtie h)(c \otimes d \bowtie k) =$
$a(h_{(1)} \cdot c) \otimes d(\tau_H(k_{(2)}) \cdot b) \bowtie h_{(2)}k_{(1)}$,

Identity: $1_A \otimes 1_A \bowtie 1_H$,

Source: $\lambda(a) = a \otimes 1_A \bowtie 1_H$,

Target: $\rho(a) = 1_A \otimes a \bowtie 1_H$,

A-A-Bimodule:
$a \cdot (b \otimes c \bowtie h) \cdot d = ab \otimes c(\tau_H(h_{(2)}) \cdot d) \bowtie h_{(1)}$

Comultiplication:
$\Delta(a \otimes b \bowtie h) = (a \otimes 1_A \bowtie h_{(1)}) \otimes_A (1_A \otimes b \bowtie h_{(2)})$,

Counit: $\varepsilon(a \otimes b \bowtie h) = a(h \cdot b)$, Antipode: $\tau(a \otimes b \bowtie h) = b \otimes a \bowtie \tau_H(h)$.

This Hopf algebroid contains the coarse Hopf algebroid A^e and the Hopf algebra H via base change $k \to A$. The total algebra contains the semidirect product $A \rtimes H$ and its derived right crossed product $H \ltimes A^{\mathrm{op}}$.

Theorem [OP]. As Hopf algebroids,
$A^e \bowtie H \cong A \odot H \odot A$
the Connes-Moscovici Hopf algebroid via

$$a \otimes b \bowtie h \longmapsto a \odot h_{(1)} \odot h_{(2)} \cdot b$$

with inverse,

$$a \odot h \odot b \longmapsto a \otimes \tau_H(h_{(2)}) \cdot b \bowtie h_{(1)}$$

GALOIS THEORY OF D2 EXTENSIONS

Action of bialgebroid $\mathrm{End}\,{}_BA_B$ on A:

$$\alpha \triangleright a = \alpha(a)$$

A becomes a left $\mathrm{End}\,{}_BA_B$-algebra, e.g.
$\alpha \triangleright (xy) = \alpha(xy) = \alpha_{(1)}(x)\alpha_{(2)}(y)$ and
$\alpha \triangleright 1_A = \lambda(\varepsilon(\alpha)) \triangleright 1_A$.
A also becomes a right $(A \otimes_B A)^B$-comodule algebra with comodule action

$$\delta(a) = a_{(0)} \otimes_R a_{(1)} = \sum_j \gamma_j(a) \otimes u_j$$

where the right D2 condition on $A\,|\,B \Rightarrow$ $\exists \gamma_j \in \mathrm{End}\,{}_BA_B$ and $u_j \in (A \otimes_B A)^B$ such that in $A \otimes_B A$ we have: $1 \otimes_B a = \sum_j \gamma_j(a) u_j^1 \otimes_B u_j^2$.

Theorem (cf. [KS]). If A_B or ${}_BA$ is faithfully flat and $A\,|\,B$ is right D2, then $A^{\mathrm{co}\,(\mathrm{A}\otimes_{\mathrm{B}}\mathrm{A})^{\mathrm{B}}} = B$ and $A \otimes_B A \cong A \otimes_R (A \otimes_B A)^B$ as A-B-bimodules, i.e., $A\,|\,B$ is a Galois extension over the bialgebroid $(A \otimes_B A)^B$.

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