The algebra of type II_1 subfactors of finite index and the Jones polynomial

Lars Kadison

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Abstract

The algebraic notions of separable extension and split extension of rings, $S \subset A$, are defined dually. If compatible sections and retracts can be found, A is said to be a finite separable extension of S. Then the endomorphism ring $\hom_S(A, A)$ is a finite separable extension of A with a special cyclic idempotent. This construction may be iterated as in Jones theory. The Jones polynomial is defined and the Jones index is shown to be the Hattori-Stallings rank of a projective module. Examples of this theory come from von Neumann algebras, fields, groups and Galois theory.

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Let R and T be rings with identity. Let ${}_T M_R$ be a T–R bimodule. M may be thought of as a generalized arrow from T into R in some extended category of rings.

Then ${}^*M = \hom({}_T M, {}_T T)$ is an R–T bimodule, since

$$(rft)m = f(mr)t$$

for every $r \in R, t \in T, m \in M$.

There is the natural map of T–T bimodules

$$\pi_M : M \otimes_R {}^*M \longrightarrow T$$

$$m \otimes f \mapsto f(m)$$

Definition 1.1 *T is M-separable over R iff π_M is a split epimorphism of T–T bimodules;*
iff there is a separability element in $M \otimes_R {}^*M$, i.e., $e = \sum_{i=1}^n m_i \otimes f_i$ satisfying

- *$te = et$ ($\forall t \in T$)*
- *$\sum_{i=1}^n f_i(m_i) = 1$.*

Now let S be a unital subring of A. Two specializations of the last definition are to be made.

First, let $M = {}_A A_S$: then $\pi_M : A \otimes_S \hom({}_A A, {}_A A) \to A$.

Definition 1.2 *A is separable over S iff*
there exists an A–A splitting for

$$\mu : A \otimes_S A \to A$$

$$a_0 \otimes a_1 \mapsto a_0 a_1$$

iff there exists $e \in A \otimes_S A : ae = ea, \forall a \in A$ and $\mu(e) = 1$.

Secondly, let $M = {}_S A_A$. Then $\pi_M : A \otimes_A \hom({}_S A, {}_S S) \to S$.

Definition 1.3 *S is M-separable over A (usually stated as "A is a split extension of S") iff*
there exists $E \in \hom_{S-S}(A, S) : E(1) = 1$.

E is called a conditional expectation for $S \subseteq A$. Note that $E(s) = s, \forall s \in S$.

You might recall that an S–S bimodule A is said to be an S-coring if there exists a coassociative comultiplication mapping $\Delta : A \to A \otimes_S A$ and bimodule morphism counit $\epsilon : A \to S$ such that $\epsilon \otimes Id \circ \Delta$ and $Id \otimes \epsilon \circ \Delta$ are the canonical isomorphisms of A onto $S \otimes_S A$ and $A \otimes_S S$.

We now define our main object of study. Now we suppose that S and A are unital K–algebras, and S continues to be a subring of A.

Definition 1.4 *A is a finite separable extension of S iff*

1. *A is a separable over S;*
2. *A is a split extension of S;*
3. *there exists a separability element e, conditional expectation E and invertible element τ in K such that $(A, S, \Delta : a \mapsto \tau^{-1}ae, \epsilon = E)$ is a coring.*

   A is said to be a finite separable extension of S with Markov trace if, in addition, there exists a trace on S, $T : S \to K$, such that

4. *$T_A = T \circ E$ is a trace on A.*

Remark 1.1 $3 \Leftrightarrow 3'$ there exists a unit $\tau \in K$, a separability element $e = \tau \sum_{i=1}^{n} x_i \otimes y_i$ and conditional expectation $E : A \to S$ such that $(\forall a \in A)$

$$\sum_{i=1}^{n} E(ax_i)y_i = a$$

and

$$\sum_{i=1}^{n} x_i E(y_i a) = a.$$

Hence,

1. the natural module ${}_SA$ is finitely generated projective with dual basis $\{y_i : i = 1, \ldots, n\}$ and $\{E(-x_i) : i = 1, \ldots, n\}$.
2. Similarly, A_S is a f. g. projective module with dual basis $x_1, \ldots, x_n$ and $E(y_1-), \ldots, E(y_n-)$.
3. E is a Frobenius homomorphism, (x_i, y_i, E) a Frobenius system, and A is a Frobenius extension of S (cf. Kasch, 1954).
4. We call τ^{-1} the *index* and denote it by $[A : S]_E$, and call (x_i, y_i, E, τ) a finite separability system for $A \supseteq S$. *We fix this notation for remainder of this paper.*

There are many good examples of finite separable extensions with Markov trace. The first that occured to me were the type II_1 subfactors of finite Jones index. But other examples are given by finite separable field extensions of degree coprime to the characteristic, group and subgroup algebras of finite index coprime to the characteristic, and tensoring an arbitrary K-algebra with trace by a strongly separable K–algebra.

Theorem 1.1 *If λ denotes the left regular representation, then* $\hom(A_S, A_S)$ *is a finite separable extension of $\lambda(A)$ with Markov trace and index τ^{-1}.*

PF. Follows from lemmas 1.1 through 1.3 below. □

Lemma 1.1 *There is a unital K-algebra structure on $A \otimes_S A$, where multiplication is given by*

$$(a_0 \otimes a_1)(a_2 \otimes a_3) = a_0 \otimes E(a_1 a_2) a_3$$

and the identity element by

$$1 = \sum_{i=1}^{n} x_i \otimes y_i = \tau^{-1} e.$$

PF. Associativity of the multiplication follows by S–bilinearity of E. The unitality of 1 follows from condition $3'$. □

Lemma 1.2 *There is a ring isomorphism, $A \otimes_S A \cong \hom(A_S, A_S)$.*

PF. The mapping is given by

$$a_0 \otimes a_1 \mapsto \lambda_{a_0} \circ E \circ \lambda_{a_1}.$$

Surjectivity: given $g \in \hom(A, A)$ and $a \in A$, we have $a = \sum_{i=1}^{n} x_i E(y_i a)$, so $g(a) = \sum g(x_i) E(y_i a)$; whence

$$\sum \lambda_{g(x_i)} \circ E \circ \lambda_{y_i} \mapsto g.$$

Injectivity is left as an exercise. □

Lemma 1.3 *The K-algebra $A \otimes_S A$ is a finite separable extension with Markov trace of the image of A under the obvious injection $a \mapsto a1$. A finite separability system is given by $(\tau^{-1} x_i \otimes 1, 1 \otimes y_i, \tau\mu, \tau)$.*

PF. Let $E_1 = \tau\mu : A \otimes_S A \to A$. Then $E_1(1) = \tau \sum x_i y_i = 1$ by definition of the separability element e.

If $T : S \to K$ is a Markov trace, then $T' = T \circ E$ is a trace on A by definition, and we must show that $T_1 = T' \circ E_1$ is a trace on $A \otimes_S A$. We compute:

$$
\begin{aligned}
T_1((a_0 \otimes a_1)(a_2 \otimes a_3)) &= T_1(a_0 E(a_1 a_2) \otimes a_3) \\
&= \tau T'(a_3 a_0 E(a_1 a_2)) \\
&= \tau T(E(a_3 a_0) E(a_1 a_2)) \\
&= \tau T'(E(a_3 a_0) a_1 a_2) \\
&= T_1((a_2 \otimes a_3)(a_0 \otimes a_1)) \square
\end{aligned}
$$

Remark 1.2 Let A_1 denote the K–algebra $A \otimes_S A$. We have just seen that A_1 is also a finite separable extension – of A – with a canonical conditional expectation, relative to which it has the same index. Also note the idempotent $1 \otimes 1 = e_1$. This is a cyclic generator of A_1 as an A–A bimodule. Also note that $T_1(e_1) = \tau$.

Now iterate the construction of A_1 for the finite separable system (A_1, A, E_1, τ). I.e., let $A_2 = A_1 \otimes_A A_1$, $E_2 = \tau\mu_2$ (the multiplication map), and $e_2 = 1 \otimes_A 1$ (where 1 is the identity in A_1). We may define A_n inductively: A_n is a finite separable extension of A_{n-1} by theorem. Note that $T_n(e_n) = \tau$, since one applies the expectation $\tau\mu_n$ and then the (normalized) trace.

After making the canonical identifications for tensors, the picture is:

$$
S \stackrel{E}{\longleftarrow} A \stackrel{E_1}{\longleftarrow} A \otimes_S A \stackrel{E_2}{\longleftarrow} \cdots \longleftarrow A_n = A^{\otimes_S n+1} \stackrel{E_{n+1}}{\longleftarrow} \cdots
$$

Proposition 1.1 *The idempotents $\{e_i : i = 1, \ldots, n\}$ satisfy the following identities in A_n:*

1. $e_i e_{i\pm 1} e_i = \tau e_i$
2. $e_i e_j = e_j e_i$ $(i - j \geq 2)$.

If there is an element t in K such that $\tau = \frac{t}{(t+1)^2}$, then the mapping

$$
\sigma_i \longmapsto w_i = (t+1)e_i - 1
$$

on the Artin generators of the braid group on $n+1$ strings, extends to a homomorphism of B_{n+1} into A_n.

PF. Nice exercise. For the second statement, one must check that $w_i w_{i+1} w_i = w_{i+1} w_i w_{i+1}$, which follows from the first statement (the braid-like relations) and the equation in K, $\tau = \frac{t}{(t+1)^2}$. One must also check the easy relation $w_i w_j = w_j w_i$. $\square$

Remark 1.3 The w_i's in A_n also satisfy the Hecke relation $w_i^2 = (t-1)w_i + t$, so that the Hecke algebra $H(t,n)$ maps into A_n in a way similar to the mapping above for braids.

Now briefly recall the two Markov moves that generate an equivalence relation on the disjoint union of all braid groups. By closing the braids to get oriented links, Markov and Alexander have shown that there is a one-to-one correspondence of ambient isotopy classes of oriented links with Markov equivalence classes of braids.

Suppose that t has a square root in K. Then given an oriented link L with braid representation $\alpha = \sigma_{i_1}^{n_1} \cdots \sigma_{i_r}^{n_r} \in B_N$, we may define the Jones polynomial of L and all isotopic links by

$$V_L(t) = (-\frac{t+1}{\sqrt{t}})^{N-1}(\sqrt{t})^{n_1+\cdots n_r} T_N(w_{i_1}^{n_1} \cdots w_{i_r}^{n_r})$$

This is actually a map from $\left(\coprod_{n\geq 2} B_n/\sim\right) \longrightarrow K$, which is an invariant of oriented links and knots. Leaving t as a variable yields a Laurent polynomial in t for knots and $\sqrt{t}$ for oriented links of two or more components. There are many more features of the link that may be read off the Jones polynomial: see [1]

Remark 1.4 The main example of the theory in this paper (announced in [3]) is a pair of von Neumann algebra II_1 factors, $N \subseteq M$, N a subfactor of finite Jones index $[M:N]$. Suppose n is the integer part of the index. It is well-known that there is a canonical condition expectation $E : M \to N$ that preserves the canonical trace on M.

M. Pimsner and S. Popa in [4] find elements $m_1, \ldots, m_{n+1} \in M$ such that these form, together with $E(m_j^* -)$, a dual basis that shows M_N is a finitely generated projective module.

The author and D. Kastler[2] have shown that

$$\frac{1}{[M:N]} \sum_{i=1}^{n+1} m_i \otimes_N m_i^*$$

is a separability element. Compatibility with E follows from certain standard formulas. In the same paper, the Jones index is shown to be the Hattori–Stallings rank $r(M)$ of the finitely generated projective module M_N (up to identification via trace of the cotrace group with scalars): we will now review and extend this in the next proposition.

Proposition 1.2 *Suppose A is a finite separable extension of S with Markov trace $T : S \to K$, such that $T(x) = 0$ iff x is a finite linear combination of commutators ($x \in [S,S]$). Then $[A : S]_E = T(r(A))$.*

PF. Let $\{y_i : i = 1, \ldots, n\}$ and $\{E(-x_i) : i = 1, \ldots, n\}$ be the dual basis of ${}_SA$ obtained from the finite separability system as before. Recall that the Hattori–Stallings rank is gotten from the dual bases and is an element in the cotrace group $S/[S,S]$. Hence,

$$T(r(A)) = T \circ E(\sum y_i x_i) = T \circ E(\sum x_i y_i) = \tau^{-1} = [A : S]_E. \square$$

References

[1] V.F.R. Jones, Subfactors and Knots, CBMS, 1991.

[2] Cohomological aspects and relative separability of finite Jones index subfactors, *Nachr. Akad. Wissen. Göttingen*, II, no. 4, 1992

[3] L. Kadison, Algebraic aspects of the Jones basic construction, C.R. Math. Reports, Canada, vol. XV, No./ 5, Oct. 1993.

[4] M. Pimsner and S. Popa, Entropy and index of subfactors, *Ann. scient. Éc. Norm. Sup.* **19** (1986), 57-106.