Subgroup Depth

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Depth Two

Frobenius extension $B \subseteq A$ depth two if

$$ {}_A A \otimes_B A_B \oplus * \cong {}_A A^q{}_B. \tag{1}$$

Suppose A and B are semisimple $\mathbb{C}$-algebras.
Let simples(A) $= \{V_1, \ldots, V_s\}$
simples(B) $= \{W_1, \ldots, W_r\}$
Apply $- \otimes_B W_i$ to Eq. (1), obtaining

$$[\mathrm{Ind}_B^A \mathrm{Res}_B^A \mathrm{Ind}_B^A W_i, V_j] \leq q[\mathrm{Ind}_B^A W_i, V_j] \tag{2}$$

Using Frobenius reciprocity, rewrite inequality:

$$\sum_{k=1}^{r} [\mathrm{Ind}_B^A W_i, \mathrm{Ind}_B^A W_k][W_k, \mathrm{Res}_B^A V_j] =$$

$$\sum_k (MM^t)_{ik} M_{kj} \leq q M_{ij}$$

if we define matrix $M_{ij} = [\mathrm{Ind}_B^A W_i, V_j]$
$= [W_i, \mathrm{Res}_B^A V_j]$, the "inclusion matrix" [6].

Summary: semisimple algebra pair $B \subseteq A$ depth two if inclusion matrix M satisfies $MM^tM \leq qM$ for some $q \in \mathbb{Z}_+$.

Theorem [7, B.K., L.K]. Suppose A and B are group algebras of finite group G and subgroup H. The extension A is D2 over $B \Leftrightarrow H \triangleleft G$.

In current preprint this is extended to

Theorem [2] Suppose A and B are semisimple algebras with B a subalgebra of A. Then A is D2 over $B \Leftrightarrow B$ is normal subring in A [12, Rieffel], i.e. every maximal ideal I in A restricts to A-invariant ideal of B: $A(I \cap B) = (I \cap B)A$.

Corollary. A depth two Hopf subalgebra K of a semisimple Hopf algebra H is normal.

Proof. Restrict kernel of counit to get condition $HK^+ = K^+H$.

Depth Three or More

Generalize depth two to tower $C \subseteq B \subseteq A$ ”depth-3” if ${}_A A \otimes_B A_C \oplus * \cong {}_A A^q{}_C$

If all semisimple algebras, $C \subseteq B$ with incl. matrix N, $B \subseteq A$ with incl. matrix $\tilde{M}$, matrix condition for depth-3 tower is similarly

$$N\tilde{M}\tilde{M}^t\tilde{M} \leq qN\tilde{M} \quad (q \in \mathbb{Z}_+)$$

Jones tower above $B \subseteq A$: incl. matrices $M, M^t, M, ..$

$B \subseteq A \hookrightarrow A_1 = \text{End}\, A_B \hookrightarrow A_2 = \text{End}\,(A_1)_A \hookrightarrow \cdots$

Theorem ([10, L.K.]) Frobenius extension $B \subseteq A$ is depth n if $B \subseteq A_{n-3} \subseteq A_{n-2}$ is depth-3 tower.

Here we have $N = M^{n-2}$, $N\tilde{M} = M^{n-1}$, and $N\tilde{M}\tilde{M}^t\tilde{M} = M^{n+1}$ where $M^2 = MM^t$, $M^3 = MM^tM$, and so on.

Corollary. Semisimple algebras $B \subseteq A$ depth n if

$$M^{n+1} \leq qM^{n-1} \tag{3}$$

for some $q \in \mathbb{Z}_+$.

Example of Depth Three: $S_2 < S_3$

S_2	(1)	(12)
η_1	1	1
η_2	1	-1

S_3	(1)	(12)	(123)
ψ_1	1	1	1
ψ_2	1	-1	1
ψ_3	2	0	-1

Induction-Restriction table:

$S_2 < S_3$	ψ_1	ψ_2	ψ_3
η_1	1	0	1
η_2	0	1	1

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad S := MM^t = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

so $MM^tM \not\leq qM$ (any $q \in \mathbb{Z}_+$),
but $S^2 \leq qS$ some q, equiv. $M^4 \leq qM^2$.

Similarly Mackey shows $S > 0$ below.

Theorem. If subgroup $H \leq G$ satisfies $H \cap xHx^{-1} = \{1_G\}$ for some $x \in G$, then depth is three.

Reapplying Mackey shows $S^n > 0$ below.

Theorem. If subgroup $H \leq G$ satisfies

$$H \cap x_1Hx_1^{-1} \cap \cdots \cap x_nHx_n^{-1} = \{1_G\}$$

for some $x_1, \ldots, x_n \in G$, then depth is $2n + 1$.

Example. $S_n < S_{n+1}$ has core

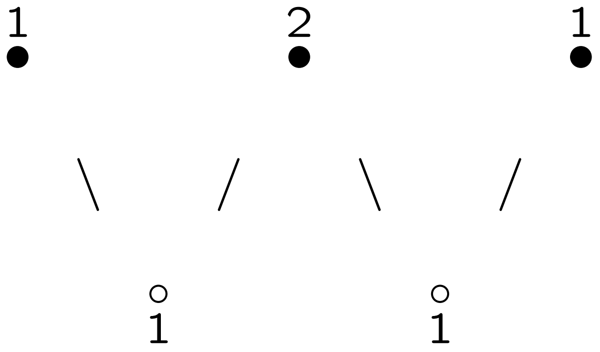
$$\{(1)\} = S_n \cap {}^{(1\ \ n+1)}S_n \cap \cdots \cap {}^{(n-1\ \ n+1)}S_n$$

so depth $2(n-1) + 1 = 2n - 1$.

Counterexample to converse: $A_5 < A_6$ has depth five, $A \cap {}^{g_1}A_5 \cap {}^{g_2}A_5 \neq \{1\}$.

Graphically, odd depth of $H < G$ is $1 +$ diameter of row of simples of H in inclusion diagram.

Example. Back to $S_2 < S_3$ with inclusion matrix $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, incidence matrix to diagram,



graphical distance between white points is two, depth is $2 + 1$.

A maximal geodesic of length $2n-2$ across the simples of S_n as subgroup of S_{n+1} in terms of partitions of n:

$$(n) \to\to (n-1,1) \to\to (n-2,1,1) \to\to \cdots$$
$$\to\to (2,1,\ldots,1) \to\to (1,\ldots,1)$$

in terms of Young diagrams for $n = 4$:

Questions of Curiosity

1. Subgroup depth takes on values of every odd number (as just seen), depth 2 (normal subgroups), and depth 4 (dihedral $D_4 < S_4$). Does subgroup depth take on values 6, 8, ...?

(It is easy graphically to realize even nos. as multimatrix subalgebra depth.)

2. Same question replacing subgroup depth with Hopf subalgebra depth (of semisimple Hopf algebra)?

3. Find a characterization of depth n subgroup (such as that for depth two in terms of normality).

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